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SURFACE DISTURBANCES IN MAGNETOHYDRODYNAMICS

By S. I. Syrovatskiy

Translation of "Nekotoryye svoystva poverkhnostey razryva v magnitnoy gidrondinamike," a dissertation submitted for the degree of Candidate of Physical and Mathematical Sciences at the P. N. Lebedev Institute of Physics of the Academy of Sciences of the USSR, November 27, 1954

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INTRODUCTION

The subject of magnetohydrodynamics deals with the study of laws governing the motion of electrically conducting liquid or gaseous media through an electromagnetic field. (For the study of gaseous media the term "magneto-gasodynamics" is frequently employed.)

The classic equations of hydrodynamics and electrodynamics are used as a starting point in the definition of magnetohydrodynamic laws. The interest in this field of studies arose only a few years ago as a result of increasing availability of data on solar atmosphere processes and also in connection with investigations on the origin of cosmic rays. Results of these investigations led to the conclusion that extensive electromagnetic fields play an essential role in the dynamics of stellar atmospheres and in phenomena occurring in interstellar space. In the meantime many more phenomena have been observed whose interpretation requires the assumption of the existence of extensive cosmic magnetic fields. At the present time, the concept of such fields is successfully employed in the explanations of the origins of cosmic radiation, polarization of light of distant stars, etc.

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Recent observations do not provide direct interstellar magnetic field data, and therefore all conclusions are based on the general laws of motion of a conducting medium in an electromagnetic field. Thus, it becomes necessary to develop a theory of magnetohydrodynamic motion. Moreover, such a theory is indispensable for understanding motion in both stellar and solar atmospheres. The latter, a highly conducting atmosphere, contains strong magnetic fields which substantially affect the character of magnetohydrodynamic motion.

The theory of magnetohydrodynamic motion may be formulated as a strictly theoretical problem, independent of its practical applications, and based on two divisions of physics, i.e. hydrodynamics and electrodynamics. It appears that motion of a conducting fluid in a magnetic field is characterized by a number of properties which are manifested the more distinctly, the more conducting the fluid. It is known that magnetic fields cannot instantaneously penetrate or "emerge" from a conductor. This is due to the presence of induction currents which impede any change of the field. In most applications of magnetohydrodynamics, the conductivity of the medium is large and the currents are attenuated slowly; as a result of this, the magnetic field appears to be "frozen" in the medium for prolonged periods of time. If, at the same time, the medium is in hydrodynamic motion, then the field is deformed with the medium. The magnetic flux through any surface formed in the moving medium will, of course, be retained. The medium in the magnetic field becomes actually anisotropic, i.e. lateral motion does not cause any changes in the field and occurs as in ordinary hydrodynamics. However, transverse motion causes deformation of the field with the accompanying transformation of the kinetic energy of the fluid into magnetic energy, or vice versa. Such transformations evolve a number of new effects which are not encountered in ordinary hydrodynamics.

Fundamental works in the field of magnetohydrodynamics are presented in Ref. [1]. Nevertheless, due to the complexity of the system of magnetohydrodynamic equations, investigation of the dynamics of conducting media is very far from completion. In particular, serious difficulties are encountered in the solution of the fundamental problem of magnetohydrodynamic turbulence, and hence there is an absence of a quantitative explanation of turbulence in a magnetic field. The above discussion points out the imperative necessity for further investigation of the subject.

The present work is devoted to the study and explanation of characteristics of surface disturbances in magnetohydrodynamics. Unusual "transient" disturbances are described. The latter exist during continuous transitions from one type of disturbance to another. Primary consideration, however, is given to the problem of stability of tangential disturbances. The problem is of considerable interest since it affords the possibility of explanation of stable and sharply defined streams, in the form of jets, bands, etc., which are observed in the atmosphere of the sun and which are difficult to explain in terms of ordinary hydrodynamics.

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*Translation of "Nekotoryye svoystva poverkhnostey razryva v magnitnoy gidrondinamike," a dissertation submitted for the degree of Candidate of Physical and Mathematical Sciences and defended at the P. N. Lebedev Institute of Physics of the Academy of Sciences of the USSR, November 27, 1954. Originally published in Trudy Fizicheskogo Instituta Akademii Nauk SSSR (Transactions of the Institute of Physics of the Academy of Sciences of the USSR), vol, 8, 1956, pp. 13-64.

In addition, investigation of the stability of tangential disturbances is important for the development of a quantitative explanation of the characteristics of magnetohydrodynamic turbulence. Turbulent motion is developed as a result of instability of laminar flow of a liquid or gas. In fully developed turbulent flow, there exists a continuous transfer of energy from larger to smaller components of the stream. The energy is finally dissipated in the smallest eddies. Instability of large eddies leads to their breakdown into smaller, also unstable eddies, which in turn break down into even smaller vortices, etc. This process continues until the viscosity of the stream prevents the occurrence of any further motion. From a theoretical point of view, the energy transfer from larger to smaller stream components is described by the non-linear terms in hydrodynamic equations, i.e. those terms which cause the instability in laminar flow. Thus, flow instability appears to be the determining factor in both the development and further course of turbulence. It is therefore of interest to investigate the stability of motion in magnetohydrodynamics.

Tangential disturbances represent the limiting case of flow with a continuously changing F
velocity gradient. The smoothing of the velocity profile results, of course, in the increase of 2
stability. Therefore, in order to establish the possibility of existence of unstable flow in magne- 3
tohydrodynamics, it is only natural to limit the investigation to a simple case of tangential disturbance. Quantitative explanation of the stabilizing action of the magnetic field as contained in this solution has a number of advantages, such as simplicity and minimum number of assumptions. Those were absent in the work of previous investigators (Refs. [2] and [3]), who studied flow stability between rotating cylinders and between parallel planes using the method of asymptotic theory of stability with many limiting assumptions. Moreover, an approach similar to the method presented herein may be extended to the case of a compressible medium which is of particular importance for practical applications.

The results obtained in Sections 5 and 6, below, verify that a sufficiently strong magnetic field stabilizes tangential disturbances in both compressible and incompressible media. Minimum values of the stabilizing field are determined in order of importance by densities of the magnetic and kinetic energies with respect to the moving medium. This represents one of the essential distinctions from ordinary hydrodynamics, where tangential disturbances are absolutely unstable. As applied to the theory of magnetohydrodynamic turbulence, the result shows that in the presence of a magnetic field, normal turbulent energy transfer is disrupted under all degrees of turbulence. The average kinetic energy of the stream may be equal to or less than the average energy of the stabilizing magnetic field. Therefore, extension of the theoretical method of locally isotropic turbulence to magnetohydrodynamic turbulence, as attempted in Refs. [4] - [6]), requires additional substantiation.

The stability of tangential disturbances and solutions of steady state magnetohydrodynamic equations, as presented in Section 9, indicate the possibility of existence of stable flow of the medium along an arbitrary magnetic field. Such flow, characterized by individual streams or jets, should under normal conditions become turbulent within a short period of time. This result is utilized in Section 10 in the interpretation of some characteristics of motion of solar protuberances.

SECTION 1. MAGNETOHYDRODYNAMIC EQUATIONS

The liquid or gaseous medium is considered to be continuous, i.e. it is assumed that the mean free path l of particles of the medium is considerably smaller than the characteristic dimension L of the phenomenon:

$$\frac{l}{L} \ll 1. \tag{1}$$

Therefore, hydrodynamic equations may be used to describe motion of the medium. These equations have to be modified by the volumetric electromagnetic force $\boldsymbol{f}_e$:

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\nabla)\,\boldsymbol{v}\right]=\boldsymbol{f}_e-\nabla p+\eta\nabla^2\boldsymbol{v}+\left(\zeta+\frac{\eta}{3}\right)\operatorname{grad}\operatorname{div}\boldsymbol{v}; \tag{2}$$

$$\frac{\partial \rho}{\partial t}+\operatorname{div}\rho\boldsymbol{v}=0, \tag{3}$$

where $\boldsymbol{v}$ = velocity of medium,
ρ = density of medium,
p = pressure,
η = viscosity,
ζ = secondary coefficient of viscosity.

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The force $\boldsymbol{f}_e$ expresses the action of the electromagnetic field on the charge and current of the medium:

$$\boldsymbol{f}_e=\rho_e\boldsymbol{E}+\frac{1}{c'}\,[\boldsymbol{jH}], \tag{4}$$

where ρ_e = density of electric charge,
$\boldsymbol{j}$ = current density,
$\boldsymbol{E}$ = vector of electric field,
$\boldsymbol{H}$ = vector of magnetic field,
c' = velocity of light.

The electromagnetic field is determined by the Maxwell equations and depends upon the charge and current of the moving medium ($\mu = 1$, $\varepsilon = \text{const}$):

$$\left.\begin{aligned}\operatorname{rot}\boldsymbol{H}&=\frac{4\pi}{c'}\boldsymbol{j}+\frac{\varepsilon}{c'}\frac{\partial \boldsymbol{E}}{\partial t},\\ \operatorname{rot}\boldsymbol{E}&=-\frac{1}{c'}\frac{\partial \boldsymbol{H}}{\partial t},\\ \operatorname{div}\boldsymbol{H}&=0,\\ \operatorname{div}\boldsymbol{E}&=\frac{4\pi}{\varepsilon}\rho_e.\end{aligned}\right\} \tag{5}$$

It is assumed that Ohm's law can be applied, and thus the current density will be equal to:

$$\boldsymbol{j}=\rho_e\boldsymbol{v}+\sigma\left(\boldsymbol{E}+\frac{1}{c'}\,[\boldsymbol{vH}]\right). \tag{6}$$

The first term of the above corresponds to the convection current and the second to the conduction current, since the charges in the field are acted upon by $\boldsymbol{E}'=\boldsymbol{E}+\frac{1}{c'}\,[\boldsymbol{vH}]$. Conductivity σ is considered to be a constant scalar quantity. The above assumption is correct if the radius of curvature R of the trajectory of electrons in the magnetic field is longer than the mean free path of the electrons l_e . Since $R=\frac{muc'}{eH}$, where u = thermal velocity of electrons, then the conductivity may be considered isotropic, if:

$$\frac{eH}{mc'}\frac{l}{u}=\omega_H\tau<1, \tag{7}$$

where ω_H = Larmor frequency,
τ = average time of free path of electrons.

When the above condition cannot be satisfied, as occurs with a highly rarefied medium in a strong magnetic field, then the conductivity may be considered anistropic (Refs. [7] and [8]); i.e. the value of conductivity will not change along the field, while it will decrease across the field $1 + \omega_H^2 \tau^2$ times. By elimination of $\boldsymbol{j}$ and $\boldsymbol{E}$ from equations (5) and (6), we obtain:

$$\frac{\partial \rho_e}{\partial t} + \operatorname{div} \rho_e \boldsymbol{v} = -\frac{4\pi\sigma}{\varepsilon} \rho_e - \frac{\sigma}{c'} \operatorname{div} [\boldsymbol{v}\boldsymbol{H}]. \tag{8}$$

Equation (8) shows that relaxation time of the charge is in the order of ε / σ. The following condition is considered to be fulfilled here:

$$\frac{4\pi\sigma L}{\varepsilon V} \gg 1, \tag{9}$$

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where V = characteristic velocity of the medium.

Condition (9) permits the convection and displacement currents to be neglected in comparison with the conducting currents[1] - which is usual in the study of electromagnetic processes in conducting media. Substituting equation (9) in equations (8) and (5), we get:

$$\rho_e = -\frac{\varepsilon}{4\pi} \operatorname{div} \left[\frac{\boldsymbol{v}}{c'} \boldsymbol{H}\right]; \tag{10}$$

$$\boldsymbol{j} = \frac{c'}{4\pi} \operatorname{rot} \boldsymbol{H}. \tag{11}$$

Substitution in equation (6) yields:

$$\boldsymbol{E} = -\frac{1}{c'} [\boldsymbol{v}\boldsymbol{H}] + \frac{c'}{4\pi\sigma} \left(\operatorname{rot} \boldsymbol{H} - \varepsilon \frac{\boldsymbol{v}}{c'} \operatorname{div} \left[\frac{\boldsymbol{v}}{c'} \boldsymbol{H}\right]\right).$$

The second term in parentheses is in the same order of magnitude as the quantity v^2 / c'^2, in comparison with the first term. Since macroscopic velocities of the medium, even in cosmic conditions, are small in comparison with the velocity of light, an accurate non-relativistic approximation can be employed by neglecting terms having the order of magnitude of v^2 / c'^2. Here,

$$\boldsymbol{E} = -\frac{1}{c'} [\boldsymbol{v}\boldsymbol{H}] + \frac{c'}{4\pi\sigma} \operatorname{rot} \boldsymbol{H}. \tag{12}$$

Also, the same approximation can be used in equation (4). Thus, using equation (11) yields:

$$\boldsymbol{f}_e = \frac{1}{4\pi} [\operatorname{rot} \boldsymbol{H} \cdot \boldsymbol{H}]. \tag{13}$$

[1] High frequency electromagnetic processes, for which $\omega \gtrsim \sigma / \varepsilon$, are not considered here.

Using above assumptions, the system of equations for a liquid of high conductivity in an electromagnetic field assumes the following form:

$$\left.\begin{aligned}
&\frac{\partial \boldsymbol{H}}{\partial t} - \operatorname{rot}\,[\boldsymbol{v}\boldsymbol{H}] = \frac{c'^2}{4\pi\sigma}\,\nabla^2 \boldsymbol{H};\\
&\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\nabla)\,\boldsymbol{v} = -\frac{1}{\rho}\,\nabla p - \frac{1}{4\pi\rho}\,[\boldsymbol{H}\cdot\operatorname{rot}\boldsymbol{H}] + \\
&\qquad\qquad + \frac{\eta}{\rho}\,\nabla^2\boldsymbol{v} + \frac{1}{\rho}\left(\zeta + \frac{\eta}{3}\right)\operatorname{grad}\operatorname{div}\boldsymbol{v};\\
&\frac{\partial \rho}{\partial t} + \operatorname{div}\rho\boldsymbol{v} = 0;\ \ \operatorname{div}\boldsymbol{H} = 0.
\end{aligned}\right\} \qquad (14)$$

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The system of equations (14) does not contain terms describing charge density, current density, and intensity of the electric field. These may be determined from equations (10) to (12). Thus, the dynamics of a conducting medium in an electromagnetic field can be fully determined by interaction between the magnetic and the velocity fields which is expressed by equations (14). For this reason, the term "magnetohydrodynamics" has been adopted to describe this phenomenon.

In cases where the compressibility of the medium cannot be neglected, equations (14) have to be augmented by the following equation of the state of the medium:

$$p = p\,(\rho,\ T). \qquad (15)$$

Equations (14) and (15) contain two vector and two scalar equations for the quantities $\boldsymbol{v}$, $\boldsymbol{H}$, p, ρ, T and must be augmented by another equation. The additional equation expresses the law of conservation of energy in the system. Since the total energy of a unit volume of the medium is equal to:

$$\frac{\rho v^2}{2} + \rho\varepsilon + \frac{H^2}{8\pi}, \qquad (16)$$

where ε = the thermal energy of the unit mass, then the equation of the conservation of energy will assume the following form:

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2} + \rho\varepsilon + \frac{H^2}{8\pi}\right) = -\operatorname{div}\boldsymbol{g}. \qquad (17)$$

The energy density g consists of the following terms: density of hydrodynamic energy flux $\rho\boldsymbol{v}\,(v^2/2 + w)$, where w = the thermal function of unit mass of electromagnetic energy flux, which is expressed by the Umov-Poynting vector $\frac{c'}{4\pi}\,[\boldsymbol{E}\boldsymbol{H}]$; the energy flux $-(\boldsymbol{v}\sigma')$, associated with the processes of internal friction, where

$$\sigma'_{ik} = \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\,\delta_{ik}\,\frac{\partial v_l}{\partial x_l}\right) + \zeta\delta_{ik}\,\frac{\partial v_l}{\partial x_l} \qquad (18)$$

(equation (18) represents the "viscous" tensor of intensity); and finally the thermal flux $-\varkappa\nabla T$, where $\varkappa$ = coefficient of thermal conductivity. Thus:

$$\boldsymbol{g} = \rho\boldsymbol{v}\left(\frac{v^2}{2} + w\right) + \frac{c'}{4\pi}\,[\boldsymbol{E}\boldsymbol{H}] - (\boldsymbol{v}\sigma') - \varkappa\nabla T.$$

Substituting equation (12) for the energy term, we get:

$$\boldsymbol{g} = \rho \boldsymbol{v}\left(\frac{v^2}{2} + w\right) + \frac{1}{4\pi}\,[\boldsymbol{H}\,[\boldsymbol{v}\boldsymbol{H}]] - \frac{\beta}{4\pi}\,[\boldsymbol{H} \operatorname{rot} \boldsymbol{H}] - (\boldsymbol{v}\sigma') - \varkappa\nabla T, \tag{19}$$

where

$$\beta = \frac{c'^2}{4\pi\sigma}. \tag{20}$$

The third term of the right-hand side of equation (19) expresses the energy flux associated with the thermal and electrical losses.

Equation (17) may be transformed by the use of equations (14) into a heat transfer equation: F 2 3

$$\rho T\left(\frac{\partial S}{\partial t} + \boldsymbol{v}\nabla S\right) = \sigma'_{ik}\frac{\partial v_i}{\partial x_k} + \frac{\beta}{4\pi}(\operatorname{rot}\boldsymbol{H})^2 + \operatorname{div}(\varkappa\nabla T), \tag{21}$$

where S = the entropy per unit mass. This equation shows that the change of quantity of heat in a moving element of volume ($dQ = \rho T\, dS$) is determined by viscosity, energy losses, and thermal conductivity.

The investigated system acts according to the law of conservation of matter (3) and the law of conservation of impulse, which may be described according to equations (14) and (18) as follows:

$$\frac{\partial \rho v_i}{\partial t} = -\frac{\partial \pi_{ik}}{\partial x_k}, \tag{22}$$

where the tensor of impulse flux density is

$$\pi_{ik} = p\delta_{ik} + \rho v_i v_k + \frac{1}{4\pi}\left(\frac{H_i^2}{2}\delta_{ik} - H_i H_k\right) - \sigma'_{ik}. \tag{23}$$

Thus, equations (14) and (21) describe the macroscopic motion of the conducting medium in an electromagnetic field, and assume the absence of convection and displacement currents. The latter assumption is true for electromagnetic processes in a highly conductive medium. The conditions of application of magnetohydrodynamic equations are expressed by inequalities (1), (7), and (9).

It follows from equations (14) that the influence of viscosity is characterized, as in ordinary hydrodynamics, by the Reynolds number $R = \frac{\rho V L}{\eta}$ (where ρ, V, and L are characteristic for a given problem and describe the density, velocity, and linear dimension of the system). The relative function of dissipation of the magnetic field due to energy losses may be expressed by the following number $R_m = \frac{4\pi\sigma L V}{c'^2}$, which represents the magnetic analogue of the Reynolds number. The studies of Ref. [9] show that in most astrophysical applications of magnetohydrodynamics, the values of R and R_m are so high that the terms for viscous dissipation and electric losses in the medium may be neglected in equations of motion without any loss of accuracy; i.e. the medium can be considered an ideal liquid with an infinite conductivity.

For an ideal medium the system of equations will be expressed:

$$\left.\begin{aligned}
&\frac{\partial \boldsymbol{H}}{\partial t} = \operatorname{rot}[\boldsymbol{v}\boldsymbol{H}];\\
&\operatorname{div} \boldsymbol{H} = 0;\\
&\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p - \frac{1}{4\pi\rho}[\boldsymbol{H} \operatorname{rot} \boldsymbol{H}];\\
&\frac{\partial \rho}{\partial t} + \operatorname{div}\rho\boldsymbol{v} = 0;\\
&\frac{\partial S}{\partial t} + \boldsymbol{v}\nabla S = 0.
\end{aligned}\right\} \tag{24}$$

It is assumed that the equation of state of the medium is known.

It is convenient to describe equations (24) in the form of laws of conservation:

$$\left.\begin{aligned}
&\frac{\partial \rho v_i}{\partial t} = -\frac{\partial \pi_{ik}}{\partial x_k};\\
&\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2} + \rho\epsilon + \frac{H^2}{8\pi}\right) = -\operatorname{div}\boldsymbol{g};\\
&\frac{\partial \rho}{\partial t} = -\operatorname{div}\rho\boldsymbol{v};\\
&\frac{\partial \boldsymbol{H}}{\partial t} = \operatorname{rot}[\boldsymbol{v}\boldsymbol{H}];\\
&\operatorname{div}\boldsymbol{H} = 0,
\end{aligned}\right\} \tag{25}$$

where

$$\pi_{ik} = p\delta_{ik} + \rho v_i v_k + \frac{1}{4\pi}\left(\frac{H^2}{2}\delta_{ik} - H_i H_k\right), \tag{26}$$

$$\boldsymbol{g} = \rho\boldsymbol{v}\left(\frac{v^2}{2} + w\right) + \frac{1}{4\pi}(H^2\boldsymbol{v} - (\boldsymbol{v}\boldsymbol{H})\boldsymbol{H}). \tag{27}$$

The first three equations in equations (25) express the laws of conservation of impulse, energy, and mass respectively. The fourth equation expresses the laws of conservation of magnetic field in a "fluid" surface that is moving together with the medium. Actually, the change of the flow of vector $\boldsymbol{H}$ through such a surface is determined by expression (10):

$$\frac{d}{dt}\iint_{(S)} \boldsymbol{H}\, d\boldsymbol{S} = \iint_{(S)}\left\{\frac{\partial \boldsymbol{H}}{\partial t} + \boldsymbol{v}\operatorname{div}\boldsymbol{H} - \operatorname{rot}[\boldsymbol{v}\boldsymbol{H}]\right\} d\boldsymbol{S},$$

where the integration is made along the liquid surface. Thus, according to equations (25),

$$\frac{d}{dt}\iint_{(S)} \boldsymbol{H}\, d\boldsymbol{S} = 0. \tag{28}$$

The above relationship is characteristic for magnetohydrodynamics by depicting the force lines of the magnetic field contained in the medium. The number of those lines in an arbitrarily moving fluid particle will, however, remain unchanged.

SECTION 2. CLASSIFICATION OF MAGNETOHYDRODYNAMIC DISTURBANCES

As in ordinary hydrodynamics, the magnetohydrodynamic equations for an ideal medium ($\eta = \zeta = \varkappa = 0$, $\sigma = \infty$) allow discontinuous solutions, where velocity, tension, or intensity of the magnetic field and other surface quantities experience step changes. In order to establish the conditions which would satisfy the solution on such surfaces, it is simple to use the laws of conservation expressed by equations (25).

A system of coordinates is chosen where the investigated element of a surface disturbance is immobile:

Let $\boldsymbol{n}$ = the vector of the normal to the surface disturbance, and
τ = an arbritrary vector in the tangential plane.

It follows directly from equations (25) that the following boundary conditions have to be satisfied:

$$\left.\begin{aligned}\{\pi_{ik}n_k\} = 0; \quad \{g_n\} = 0; \quad \{\rho v_n\} = 0;\\ \{[\boldsymbol{vH}]_\tau\} = 0; \quad \{H_n\} = 0.\end{aligned}\right\} \tag{29}$$

In condition equations (29) and hereafter, the braces will denote the difference of quantities contained therein on both sides of the surface disturbance. The meaning of conditions (29) is evident: the first three express the continuity of momentum, energy and mass fluxes; the next two represent simple electrodynamic conditions of continuity of tangential and normal components of the electric and magnetic fields, respectively. Therefore, with $\sigma = \infty$ from equation (12), it follows that:

$$\boldsymbol{E} = -\frac{1}{c'}[\boldsymbol{vH}]. \tag{30}$$

If the x axis of the coordinate system is directed along the normal to the surface, then boundary equations (29) can be written as:

$$\left.\begin{gathered}\left\{p + \rho v_x^2 + \frac{H^2}{8\pi}\right\} = 0;\\ \left\{\rho v_x v_y - \frac{1}{4\pi}H_xH_y\right\} = 0; \quad \left\{\rho v_x v_z - \frac{1}{4\pi}H_xH_z\right\} = 0;\\ \left\{\rho v_x\left(\frac{v^2}{2} + w\right) + \frac{1}{4\pi}[H^2v_x - (\boldsymbol{vH})H_x]\right\} = 0;\\ \{\rho v_x\} = 0; \quad \{H_x\} = 0;\\ \{v_xH_y - v_yH_x\} = 0; \quad \{v_xH_z - v_zH_x\} = 0.\end{gathered}\right\} \tag{31}$$

Axes y and z. were selected as two independent directions of the vector τ .

In ordinary hydrodynamics there are possible two mutually exclusive types of disturbances: tangential and normal (shock wave). A continuous transition between these two types is impossible. Therefore, during perturbation of surface disturbances, their classification will not change and, as in ordinary hydrodynamics, will have real physical significance. The characteristic features of magnetohydrodynamic disturbances are given below. With a continuous change of the conditions of motion, any surface disturbance allowable by equations (31) can be transformed into any other type, as will be shown later. Therefore, the type of classification of disturbances used in ordinary hydrodynamics cannot be employed here. All disturbances are interrelated by transitions and, in this sense, form one general type. Nevertheless, for the purpose of expediency, a conditional classification of magnetohydrodynamic disturbances is presented. This classification basically corresponds to the classification introduced in Ref. [11], which in turn is based on external indications in the vicinity of the disturbance.

Tangential disturbance. Analogous with ordinary hydrodynamics, surface disturbances with an absence of normal component of velocity are included in this category, thus:

$$v_x = 0. \tag{32}$$

If the normal component of field H_x differs from zero, then, according to equation (31), the velocity, pressure, and magnetic field should be continuous. Such disturbance represents stable boundary between two different media. It is assumed that

$$H_x = 0. \tag{33}$$

In this case the magnetic field and velocity are parallel to the surface of the disturbance and according to conditions (31) can undergo arbitrary changes of both magnitude and direction. Step changes of pressure should be related to step changes of the intensity of magnetic field by the following condition:

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$$\left\{p + \frac{H^2}{8\pi}\right\} = 0. \tag{34}$$

Conditions (32) to (34) fully characterize the magnetohydrodynamic tangential surface disturbance. Such surface disturbances are possible in incompressible as well as compressible media.

Perpendicular shock wave. Disturbances for which:

$$v_x \neq 0, \quad H_x = 0, \tag{35}$$

are classified as perpendicular shock waves. It follows from equations (31) that the tangential component of velocity is continuous, or:

$$\{v_y\} = 0, \quad \{v_z\} = 0, \tag{36}$$

and that the tangential component of field intensity satisfies the following conditions:

$$\{v_x H_y\} = 0, \quad \{v_x H_z\} = 0. \tag{37}$$

Conditions (36) and (37) permit the transformation to a system of coordinates where both sides of the disturbance are:

$$v_y = 0, \quad v_z = 0, \quad H_z = 0. \tag{38}$$

In this system of coordinates, and from conditions (31), (35), and (38), the boundary equations for a perpendicular shock wave will have the following limits ($v_x = v,\ H_y = H$):

$$\left.\begin{array}{c} \left\{\dfrac{H}{\rho}\right\} = 0; \ \{\rho v\} = 0; \\ \left\{\dfrac{v^2}{2} + w + \dfrac{H^2}{4\pi\rho}\right\} = 0, \ \left\{p + \rho v^2 + \dfrac{H^2}{8\pi}\right\} = 0. \end{array}\right\} \tag{39}$$

Surface disturbance of this type represents a longitudinal shock wave whose direction of propagation is perpendicular to the direction of the magnetic field as shown in Figure 1. With $H = 0$, this is a simple shock wave. With $H \neq 0$, the magnetic field is diminishing the compressibility of the medium and correspondingly enlarges the velocity of propagation of the surface disturbance. The latter was listed in Refs. [12] and [13].

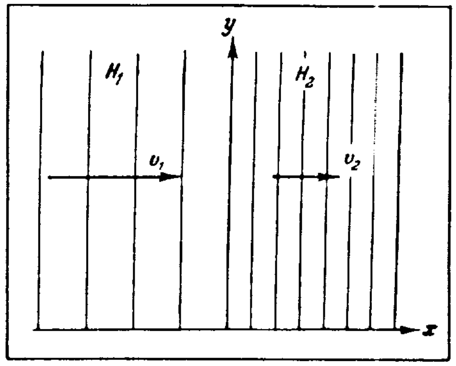

Fig. 1. Longitudinal shock wave.

The first of conditions (39) expresses the "attachment" of the magnetic field to the matter of the medium: the quantity H/ρ is preserved. The remaining equations will have the form

$$\varepsilon^{*} = \varepsilon + \frac{H^{2}}{8\pi\rho}, \quad p^{*} = p + \frac{H^{2}}{8\pi} \tag{40}$$

and will assume the form of simple equations for shock waves, where the energy and pressure depend upon the intensity of the field according to formulae (40). As shown by Kaplan and Stanyukovich, Ref. [14], the problem of arbitrary unidimensional magnetohydrodynamic flow through a perpendicular field can be reduced to a simple hydrodynamic problem with corresponding changes in the equation of state. In particular, this pertains to a perpendicular wave.

For the investigation of disturbances with

$$v_x \neq 0 \text{ and } H_x \neq 0, \tag{41}$$

it is convenient to use a system of coordinates where the vectors $\boldsymbol{v}$ and $\boldsymbol{H}$ are parallel on one side of the disturbance. When conditions (41) are satisfied, this is always possible. Actually it is sufficient to change to a system of coordinates whose origin moves parallel to the surface disturbance with the following velocity:

$$\boldsymbol{V} = \boldsymbol{v} - \frac{v_x}{H_x}\boldsymbol{H}. \tag{42}$$

The above equations hold true in cases where $\boldsymbol{v}$ and $\boldsymbol{H}$ are equal to the velocity and intensity of the field in the original system of coordinates. It follows from boundary equations (31) that in this system of coordinates, vectors $\boldsymbol{v}$ and $\boldsymbol{H}$ will also be parallel on the other side of the disturbance. By indexing the two sides of the disturbance by numbers 1 and 2 where ($x < 0$ and $x > 0$, respectively), the condition may be written as:

$$\boldsymbol{v}_1 = q_1\boldsymbol{H}_1, \quad \boldsymbol{v}_2 = q_2\boldsymbol{H}_2, \tag{43}$$

where q_1 and q_2 are coefficients of proportionality. In the selected system of coordinates, the lines of flow of the liquid (gas) are parallel to the magnetic force lines and undergo similar changes and breaks on the surface of the disturbance. It should be noted that for a perpendicular shock wave, and also in general cases of tangential disturbances, the selection of a coordinate system where the motion takes place according to conditions (43) is impossible.

With conditions (41) and (43), the boundary equations (31) assume the following form:

$$\left.\begin{array}{c}\left\{p+\rho v_x^2+\frac{1}{8\pi}(H_y^2+H_z^2)\right\}=0;\ \left\{w+\frac{v^2}{2}\right\}=0;\\ \left\{\rho v_x v_y-\frac{1}{4\pi}H_xH_y\right\}=0;\ \left\{\rho v_x v_z-\frac{1}{4\pi}H_xH_z\right\}=0;\\ \{\rho v_x\}=0;\quad \{H_x\}=0.\end{array}\right\} \tag{44}$$

From the above equations and relationships (43), the following equations may be formed:

$$\{\rho q\}=0; \tag{45}$$

$$\left\{\left(1-\frac{1}{4\pi\rho q^2}\right)v_y\right\}=0;\ \left\{\left(1-\frac{1}{4\pi\rho q^2}\right)v_z\right\}=0. \tag{46}$$

Equations (44) and equations (45) and (46) assume two different types of motion depending on whether there exists a continuity or a step-change of the density of the medium on the surface of the disturbance.

Magnetohydrodynamic wave. It is assumed that despite conditions (41) the following condition of continuity of density is true:

$$\{\rho\}=0. \tag{47}$$

Equations (44) and (45) and condition (46) lead to the continuity of the normal velocity component and of the coefficient of proportionality q between vectors of the magnetic field and velocity:

$$\{v_x\}=0;\quad \{q\}=0. \tag{48}$$

Equations (46) may be reduced as follows:

$$\left(q^2-\frac{1}{4\pi\rho}\right)\{v_y\}=0;\ \left(q^2-\frac{1}{4\pi\rho}\right)\{v_z\}=0.$$

If at least one of the quantities $\{v_y\}$ or $\{v_z\}$ is other than zero, then:

$$q=\pm\frac{1}{\sqrt{4\pi\rho}}. \tag{49}$$

Otherwise, in accordance with equations (43), (47), and (48), disturbance will be formed. Thus, in magnetohydrodynamic waves, the velocity vector is related to the magnetic field vector by the following relationships:

$$\boldsymbol{v}_1=\pm\frac{1}{\sqrt{4\pi\rho}}\boldsymbol{H}_1,\ \boldsymbol{v}_2=\pm\frac{1}{\sqrt{4\pi\rho}}\boldsymbol{H}_2. \tag{50}$$

In accordance with equations (44), (47), and (48), the following boundary equations will be satisfied on the surface:

$$\left.\begin{aligned} \{v_x\} = 0; \quad \{H_x\} = 0; \\ \{s\} = 0; \quad \{\rho\} = 0; \end{aligned}\right\} \tag{51}$$

$$\left\{p + \frac{1}{8\pi}(H_y^2 + H_z^2)\right\} = 0, \tag{52}$$

the expression $w = \varepsilon + p/\rho$ is used here as the thermal function per unit mass of the medium.

In accordance with conditions (51) and (52), the following note should be made. Since the density and internal energy of the medium is equal on both sides of the disturbance, then other thermodynamic quantities, such as pressure, should also be equal. This means that for a medium with a unique equation of state the following two conditions will have to be satisfied:

$$\{p\} = 0, \quad \{H_y^2 + H_z^2\} = 0, \tag{53}$$

i.e. the surface disturbance, despite continuity of all thermodynamic quantities, can be characterized by conditions (50) and by the continuity of normal components and absolute quantities of velocity and intensity of the magnetic field. With known values of $\boldsymbol{H}_1$ (and thus $\boldsymbol{v}_1$), there are possible values of $\boldsymbol{H}_2$ (and thus v_2) which are located on a surface of a cone, the resultant of which forms an angle with the normal equal to the angle of obliquity of vector $\boldsymbol{H}_1$. The character of the motion in such a disturbance is shown on Figure 2.

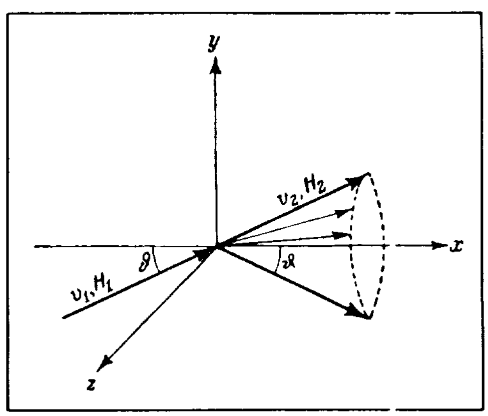

Fig. 2. Magnetohydrodynamic wave.

In Ref. [11], an analogous surface disturbance for a relativistic problem was called a "Symmetric Shock Wave".

In cases of incompressible fluids, where thermodynamic correspondences are not considered, conditions (53) do not follow from boundary equations. Tangential components of field intensity and corresponding velocities may undergo arbitrary step changes related to step changes of pressure by equation (52). This means that in an incompressible liquid, minute changes of density and internal energy may lead to finite changes in pressure. Therefore, conditions $\{\rho\} = 0$ and $\{\varepsilon\} = 0$ may be considered satisfied when the step-change of pressure is different from zero and balances the difference of magnetic intensity on both sides of the surface disturbance.

The velocity of propagation of magnetohydrodynamic waves is determined directly by expressions (50): since the normal component of velocity is continuous and equal to $v_n = \pm H_n/\sqrt{4\pi\rho}$, then relatively to the medium the surface disturbance is propagated with the velocity $\mp H_n/\sqrt{4\pi\rho}$.

The chief characteristic of magnetohydrodynamic waves is the possibility of transmission of a tangential momentum to the medium. This is due to the fact that the motion in general cases is not planar. The solution to the problem of magnetohydrodynamic waves was first obtained by Ref. [7].

Inclined shock wave. Disturbances of this type, despite conditions (41), are characterized by the presence of step-change in density:

$$\{\rho\} \neq 0. \tag{54}$$

In addition the motion should be unidirectional, i.e. a transition to a system of coordinates may be made where the motion takes place in surfaces (x, y) and

$$v_{1z} = 0, \quad H_{1z} = 0, \quad v_{2z} = 0, \quad H_{2z} = 0. \tag{55}$$

Rotation of the coordinate system about its x axis to give $v_{1z} = 0$, and therefore $H_{1z} = 0$, can always be made. According to conditions (46), either $v_{2z} = 0$ or $H_{2z} = 0$ leads directly to conditions (55), since

$$q_2^2 = \frac{1}{4\pi\rho_2}; \quad q_1^2 \neq \frac{1}{4\pi\rho}. \tag{56}$$

(Simultaneous equations $q_2^2 = \frac{1}{4\pi\rho_2}$ and $q_1^2 = \frac{1}{4\pi\rho_2}$ are incompatible with conditions (45) and (54)). It follows from the last case of equations (46) that $v_{1y} = 0$ and, therefore, $H_{1y} = 0$, i.e. on one side of the disturbance the tangential components of velocity and field intensity are generally absent and the system of coordinates has to be selected so as to make $v_{2z} = 0$ and $H_{2z} = 0$.

In the selected system of coordinates according to conditions (55), the motion is characterized as shown on Figure 3, and the boundary equations (44) may be brought to the following:

$$\left.\begin{aligned} &\left\{p + \rho v_x^2 + \frac{H_y^2}{8\pi}\right\} = 0; \ \left\{w + \frac{v^2}{2}\right\} = 0; \\ &\{H_x\} = 0; \ \{\rho v_x\} = 0; \ \left\{\rho v_x v_y - \frac{1}{4\pi} H_x H_y\right\} = 0. \end{aligned}\right\} \tag{57}$$

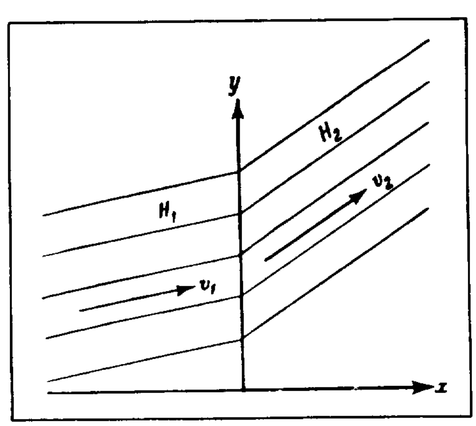

Fig. 3. Inclined shock wave.

In disturbances of this type, the shock wave interacts in a complicated manner with the magnetic field. The propagation velocity of an inclined shock wave depends not only on the degree of compressibility, but also on the direction and magnitude of the magnetic field. The dependence between parameters which determine the state of the medium prior to and after passing of the shock wave was shown in Refs. [12] and [13].

In particular, when $H_y = 0$ on both sides of the disturbance, and therefore $v_y = 0$, i.e. the disturbance is propagated along the magnetic field, the latter does not influence the propagation of the shock wave. This "parallel" shock wave may be described by simple hydrodynamic equations:

$$\{\rho v_x\} = 0; \quad \left\{w + \frac{v^2}{2}\right\} = 0; \quad \{p + \rho v_x^2\} = 0. \tag{58}$$

For a disturbance which follows the condition shown in equation (56), the system of equations (44) may be reduced to the following:

$$\left.\begin{aligned} &v_{1y} = 0; \quad H_{1y} = 0; \quad v_{2y} = \pm \frac{H_{2y}}{\sqrt{4\pi\rho_2}}; \\ &\left\{w + \frac{v^2}{2}\right\} = 0; \quad \{\rho v_x\} = 0; \quad \{H_x\} = 0; \\ &p_1 + \rho_1 v_{1x}^2 = p_2 + \rho_2 v_{2x}^2 + \frac{\rho_2 v_{2y}^2}{2}, \end{aligned}\right\} \tag{59}$$

and the flow is characterized as shown in Figure 4. One side of the surface, the tangential components of magnetic field and velocity are absent and motion takes place in the same manner as in a parallel shock wave. On the other side of the disturbance, motion is characterized by a magnetohydrodynamic wave, where there is a possibility of the existence of arbitrary tangential components of the magnetic field and velocity.

Among the described surface disturbances, there is an absence of flow of matter through the surface, and therefore those are tangential surface disturbances. Conversely, in disturbances of the three following types, matter is transferred through the surface or the disturbance is propagated relatively to the medium. Therefore, such disturbances are called waves of one or another type. The velocity of propagation in the limiting case of disturbances of low intensity, i.e. in a case analogous to the propagation of sound waves in ordinary hydrodynamics, may be found in Refs. [15] to [17].

It was shown that in an incompressible medium there are only two types of disturbances possible: namely, the tangential and the magnetohydrodynamic wave.

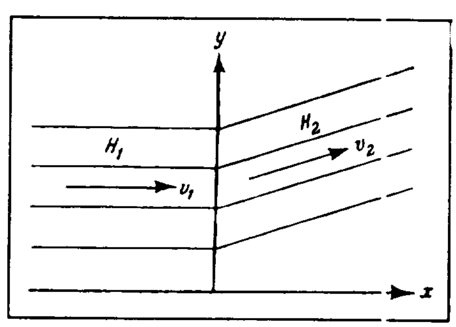

Fig. 4. Disturbance according to:

$$q_2^2 = \frac{1}{4\pi\rho_2}, \quad q_1^2 \neq \frac{1}{4\pi\rho_1}.$$

It can now be shown that in magnetohydrodynamics, transitions of surface disturbances of one type into disturbances of another type are possible. It is sufficient to establish that there are disturbances (called "transition" disturbances) that may be simultaneously classified in two different types. Therefore, with the relatively small change of parameters, a change from one to another neighboring type may occur. The transient disturbances may be found by direct comparison of the boundary equations which relate disturbances of two types. The values of parameters, where the boundary conditions for two different types of disturbances correspond to each other may determine the transition between two types. The possibility of the existence of a continuous transition between disturbances of two types will be investigated below.

Comparing conditions (32) to (34) with conditions (36) and (39) it is found that for a continuous transition from a tangential disturbance to a perpendicular shock wave it is necessary to have a disturbance that is characterized by continuous velocity and the following conditions:

$$\left\{\frac{H_y}{\rho}\right\} = 0, \quad \left\{w + \frac{H_y^2}{4\pi\rho}\right\} = 0, \quad \left\{p + \frac{H_y^2}{8\pi}\right\} = 0,$$

i.e. the disturbance affects only the thermodynamic quantities and the tangential component of the field intensity.

Such a disturbance, however, is impossible. The above conditions require the continuity of three independent functions of the three variables H_y, p and ρ. This means that the above variables should be the same on both sides of the disturbance, i.e. the discontinuity is not present. Thus, direct transition between a tangential disturbance and a perpendicular shock wave is impossible.

Comparison of equations (32) to (34) with conditions (50) to (52) shows that the disturbance satisfies the following conditions:

$$\left.\begin{aligned} &v_x = 0, \quad H_x = 0, \quad \{\varepsilon\} = 0, \quad \{\rho\} = 0, \\ &\left\{p + \frac{H^2}{8\pi}\right\} = 0, \quad \{\boldsymbol{v}\} = \pm\frac{1}{\sqrt{4\pi\rho}}\{\boldsymbol{H}\}. \end{aligned}\right\} \tag{60a}$$

The above applies to the transition between a tangential and a magnetohydrodynamic wave. As noted, in a compressible medium, with pressure as a unique function of density and internal energy, equations (60a) are equivalent to the following:

$$\left.\begin{aligned} &v_x = 0; \quad H_x = 0; \quad \{\varepsilon\} = 0; \quad \{\rho\} = 0; \quad \{p\} = 0; \\ &\{H_y^2 + H_z^2\} = 0; \qquad \{\boldsymbol{v}\} = \pm\frac{1}{\sqrt{4\pi\rho}}\{\boldsymbol{H}\}. \end{aligned}\right\} \tag{60b}$$

Thus, a continuous transition between a tangential disturbance and a magnetohydrodynamic wave in an incompressible medium is possible if conditions (60a) are satisfied. For a compressible medium, the same applies to conditions (60b).

In an inclined shock wave, the motion is planar and therefore the transition to it is possible only from a planar tangential disturbance ($v_z = 0$, $H_z = 0$). Here, the condition of parallelism of vectors $\boldsymbol{v}$ and $\boldsymbol{H}$ does not determine the coordinate system which was used for an inclined shock wave [see equations (32), (33), and (42)]. Direct application of the original boundary conditions, as shown in equations (31), must be made. These equations may be written as follows:

$$\left\{p + \rho v_x^2 + \frac{H_y^2}{8\pi}\right\} = 0;$$

$$\rho v_x \left\{\frac{v^2}{2} + w + \frac{H_y^2}{4\pi\rho}\right\} = \frac{1}{4\pi} H_x \{vH\};$$

$$\rho v_x \{v_y\} = \frac{1}{4\pi} H_x \{H_y\}; \quad \rho v_x \left\{\frac{H_y}{\rho}\right\} = H_x \{v_y\}.$$

The above relationships must be fulfilled in an inclined shock wave up to and including $\rho v_x = 0$ and $H_x = 0$. Excluding the latter, it will be found from the above three equations that limiting relationships which have to be satisfied for a transition between a tangential an inclined shock-wave are (assuming that $v_x = 0$ and $H_x = 0$):

$$\left.\begin{aligned} v_x = H_x = 0; \quad \left\{p + \frac{H_y^2}{8\pi}\right\} = 0; \\ \{v_y\}^2 = \frac{1}{4\pi}\left\{\frac{H_y}{\rho}\right\}\{H_y\}; \\ \left\{\frac{v_y^2}{2} + w + \frac{H_y^2}{4\pi\rho}\right\}\{H_y\} = \{v_y H_y\}\ \{v_y\}. \end{aligned}\right\} \quad (61)$$

It is easy to show that direct transition between a perpendicular shock wave and a magnetohydrodynamic wave is impossible. Such transition could take place only when $\rho = 0$, but conditions (39) show that in such a state disturbance cannot exist.

The transition between perpendicular and inclined shock waves is always possible. Such transition will be caused by the appearance or disappearance of the normal component of field intensity. Thus, the perpendicular shock wave is simply a singular case of inclined shock wave. The separation of these two types of waves into different categories may be justified by the simplicity of the perpendicular shock wave and by the fact that in the case of a perpendicular shock wave it is impossible to introduce a special coordinate system, where vectors v and H are parallel as in equation (43) and which is characteristic for inclined waves.

Finally the inclined shock wave, where the step-change of density approaches zero, changes into a planar magnetohydrodynamic wave. Since the planar magnetohydrodynamic wave, in a compressible medium, should satisfy the condition $\{H_y^2\} = 0$, then either $H_{y_1} = H_{y_2}$ or $H_{y_1} = -H_{y_2}$. In the first case all quantities are continuous and there is an absence of discontinuity.

Therefore, the transition discontinuity between inclined and magnetohydrodynamic waves is represented by a planar "symmetrical" discontinuity (shown in Figure 5) which satisfies conditions (51) and (53):

$$\left.\begin{aligned} \{v_x\} = 0; \quad \{H_x\} = 0; \quad \{\varepsilon\} = 0; \quad \{p\} = 0; \quad \{\rho\} = 0; \\ v = \pm \frac{H}{\sqrt{4\pi\rho}}; \quad H_{y_1} = -H_{y_2}. \end{aligned}\right\} \quad (62)$$

In this all quantities are continuous with the exception of the tangential components of field and velocity. The latter change their direction on the surface of the disturbance.

In an incompressible liquid there are possible only two types of disturbances: tangential and magnetohydrodynamic wave. Figure 6 shows the possible transitions between disturbances of different types.

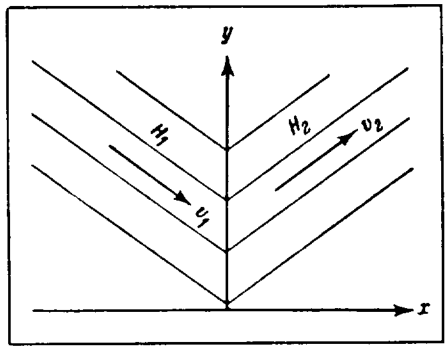

Fig. 5. Planar symmetrical disturbance.

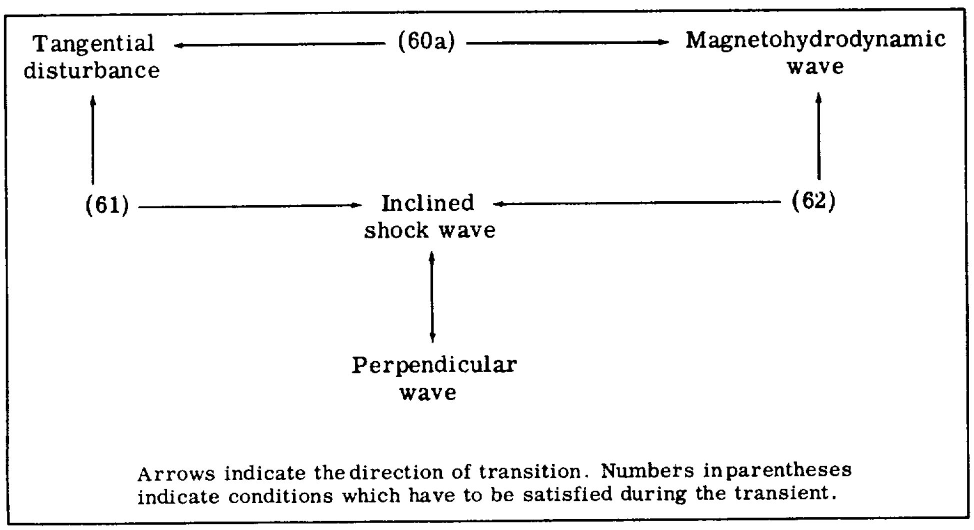

Fig. 6. Possible transitions among disturbances of different types.

SECTION 4. PERTURBATIONS OF SURFACE DISTURBANCES

Boundary equations are derived to satisfy small perturbations of steady state surface disturbances. Assume that a steady state surface disturbance, where $x = 0$, assumes the following form as a result of perturbation:

$$\Phi(x, y, z, t) \equiv x - \xi(y, z, t) = 0. \tag{63}$$

A new set of coordinates is introduced, at an arbitrary point y, z of this surface. This new system moves parallel to the origin along the x axis together with the arbitrary point on the surface. Velocity and intensity of the magnetic field will be expressed in the new system by the following relationships:

$$\boldsymbol{v}^* = \boldsymbol{v} - \boldsymbol{i}\frac{\partial \xi}{\partial t}, \quad \boldsymbol{H}^* = \boldsymbol{H}, \tag{64}$$

where $\boldsymbol{i}$ is the only vector of the x axis. In this system o coordinates, the element of the surface is immobile (rotation of a small segment of the surface may be neglected) and the boundary conditions will assume the following form [refer to equation (29)]:

$$\left.\begin{aligned}&\{\boldsymbol{H}^{\bullet}\boldsymbol{n}\} = 0; \quad \{\rho\boldsymbol{v}^{\bullet}\boldsymbol{n}\} = 0; \qquad \{\boldsymbol{g}^{\bullet}\boldsymbol{n}\} = 0;\\ &\{\overset{\bullet}{\pi}_{ik}n_k\} = 0; \quad \{[\boldsymbol{v}^{\bullet}\boldsymbol{H}^{\bullet}]\,\tau_1\} = 0; \quad \{[\boldsymbol{v}^{\bullet}\boldsymbol{H}^{\bullet}]\,\tau_2\} = 0 \quad \text{при } x = \xi.\end{aligned}\right\} \tag{65}$$

In the above equations $\overset{\bullet}{\pi}_{ik}$ and $\boldsymbol{g}^{\bullet}$ are expressed by $\boldsymbol{v}^{\bullet}$ and $\boldsymbol{H}^{\bullet}$ in an ordinary manner according to formulae (26) and (27). Since the normal to the perturbed surface equals:

$$\boldsymbol{n} = \nabla\Phi = \left\{1,\ -\frac{\partial\xi}{\partial y},\ -\frac{\partial\xi}{\partial z}\right\},$$

and the following may be taken as the two independent tangents

$$\tau_1 = \left\{\frac{\partial\xi}{\partial y},\ 1, 0\right\},$$

$$\tau_2 = \left\{\frac{\partial\xi}{\partial z},\ 0, 1\right\},$$

then equation (65) may be rewritten as:

$$\left.\begin{aligned}&\left\{H_x^{\bullet} - H_y^{\bullet}\frac{\partial\xi}{\partial y} - H_z^{\bullet}\frac{\partial\xi}{\partial z}\right\} = 0;\\ &\left\{\rho\left(v_x^{\bullet} - v_y^{\bullet}\frac{\partial\xi}{\partial y} - v_z^{\bullet}\frac{\partial\xi}{\partial z}\right)\right\} = 0;\\ &\left\{g_x^{\bullet} - g_y^{\bullet}\frac{\partial\xi}{\partial y} - g_z^{\bullet}\frac{\partial\xi}{\partial z}\right\} = 0;\\ &\left\{\overset{\bullet}{\pi}_{ix} - \overset{\bullet}{\pi}_{iy}\frac{\partial\xi}{\partial y} - \overset{\bullet}{\pi}_{iz}\frac{\partial\xi}{\partial z}\right\} = 0.\\ &\left\{[\boldsymbol{v}^{\bullet}\boldsymbol{H}^{\bullet}]_x\frac{\partial\xi}{\partial y} + [\boldsymbol{v}^{\bullet}\boldsymbol{H}^{\bullet}]_y\right\} = 0\\ &\left\{[\boldsymbol{v}^{\bullet}\boldsymbol{H}^{\bullet}]_x\frac{\partial\xi}{\partial z} + [\boldsymbol{v}^{\bullet}\boldsymbol{H}^{\bullet}]_z\right\} = 0\end{aligned}\right\} \tag{66}$$

when $x = \xi$.

Referring to the original system of coordinates according to equation (64), boundary equations for an arbitrarily deformed surface are easily obtained.

It is further assumed that perturbation of the original steady state motion and corresponding surface is small. In other words, the perturbation may be described by the following quantities:

$$\boldsymbol{v} + \boldsymbol{v}',\ \boldsymbol{H} + \boldsymbol{h},\ p + p',\ \rho + \rho',\ \ldots \tag{67}$$

where $\boldsymbol{v}$, $\boldsymbol{H}$, p and ρ are characteristic for steady state motion and satisfy boundary equations (31) on the surface when $x = 0$, and $\boldsymbol{v}'$, $\boldsymbol{h}$, p' and ρ' are small perturbations which may be neglected. Since the shift of the surface ξ is a quantity of the same order of magnitude as the perturbation, then boundary conditions may be applied to a plane where $x = 0$ instead of $x = \xi$. Using assumptions from equations (64) and (66) and substitution of expressions (67) and (31), the following system of boundary conditions for perturbations may be derived:

$$\left.\begin{aligned}
&\{h_n\}=0; \quad \{\rho v_n' + \rho' v_x\} = 0;\\
&\Big\{\Big(\frac{v^2}{2}+w+\frac{H^2}{4\pi\rho}\Big)\rho v_n' - \frac{1}{4\pi}(\boldsymbol{vH})\,h_n - \frac{1}{4\pi}H_x(\boldsymbol{vh}+\boldsymbol{Hv}') +\\
&+ v_x\Big[\rho'\Big(\frac{v^2}{2}+w\Big)+\rho\Big(\bar{\boldsymbol{v}}\bar{\boldsymbol{v}}' - v_x\frac{\partial\xi}{\partial t}+w'\Big)+\frac{1}{2\pi}(\boldsymbol{Hh})\Big]\Big\}=0;\\
&\Big\{p' + \rho' v_x^2 + \rho v_x v_x' + \rho v_x v_n' + \frac{1}{4\pi}(H_y h_y + H_z h_z)\Big\}=0;\\
&\Big\{\rho v_y v_n' + \rho' v_y v_x + \rho v_x\Big(v_y' + v_x\frac{\partial\xi}{\partial y}\Big) - \frac{1}{4\pi}(H_x h_y + H_y h_n)\Big\}=0;\\
&\Big\{\rho v_z v_n' + \rho' v_z v_x + \rho v_x\Big(v_z' + v_x\frac{\partial\xi}{\partial z}\Big) - \frac{1}{4\pi}(H_x h_z + H_z h_n)\Big\}=0;\\
&\{v_y h_n - H_y v_n' - v_x h_y + v_y' H_x\}=0;\\
&\{v_z h_n - H_z v_n' - v_x h_z + v_z' H_x\}=0
\end{aligned}\right\} \tag{68}$$

when $x=0$. In the above equations

$$\left.\begin{aligned}
v_n' &\equiv v_x' - \frac{\partial\xi}{\partial t} - v_y\frac{\partial\xi}{\partial y} - v_z\frac{\partial\xi}{\partial z},\\
h_n &\equiv h_x - H_y\frac{\partial\xi}{\partial y} - H_z\frac{\partial\xi}{\partial z},
\end{aligned}\right\} \tag{69}$$

denote normal components of perturbations of velocity and intensity of the magnetic field.

Equations (68), applicable to surface disturbances of various types, permit determination of transitions by different means than were used in the previous section. Perturbations of a tangential disturbance are investigated below as an example. The results of this investigation are further used in the study of stability of a tangential disturbance with relation to small perturbations. In the case of a tangential disturbance, the steady state motion satisfies conditions (32) to (34):

$$v_x=0, \quad H_x=0, \quad \Big\{p+\frac{H^2}{8\pi}\Big\}=0, \tag{70}$$

and the system of boundary equations (6) may be reduced to the following:

$$\left.\begin{aligned}
&\{h_n\}=0; \qquad \{\rho v_n'\}=0;\\
&\Big\{p' + \frac{1}{4\pi}(H_y h_y + H_z h_z)\Big\}=0;\\
&\Big\{\Big(\frac{v^2}{2}+w+\frac{H^2}{4\pi\rho}\Big)\rho v_n' - \frac{1}{4\pi}(\boldsymbol{v}\,\boldsymbol{H})\,h_n\Big\}=0;\\
&\Big\{\rho v_y v_n' - \frac{1}{4\pi}H_y h_n\Big\}=0; \quad \Big\{\rho v_z v_n' - \frac{1}{4\pi}H_z h_n\Big\}=0;\\
&\{v_y h_n - H_y v_n'\}=0; \qquad \{v_z h_n - H_z v_n'\}=0.
\end{aligned}\right\} \tag{71}$$

Since the quantities h_n and $\rho v_n'$ are continuous, the last five equations may be rewritten:

$$\left.\begin{aligned}
&\Big\{\frac{v^2}{2}+w+\frac{H^2}{4\pi\rho}\Big\}\rho v_n' = \frac{1}{4\pi}\{\boldsymbol{vH}\}\,h_n;\\
&\{v_y\}\,\rho v_n' = \frac{1}{4\pi}\{H_y\}\,h_n; \qquad \Big\{\frac{H_y}{\rho}\Big\}\rho v_n' = \{v_y\}\,h_n;\\
&\{v_z\}\,\rho v_n' = \frac{1}{4\pi}\{H_z\}\,h_n; \qquad \Big\{\frac{H_z}{\rho}\Big\}\rho v_n' = \{v_z\}\,h_n.
\end{aligned}\right\} \tag{72}$$

When parameters of a nonperturbed surface are known, the system of equations (72) contains five equations for two unknown quantities $\rho v_n'$ and h_n. Since the system is over-determined, only a trivial solution is possible: $h_n = 0$ and $\rho v_n = 0$. Thus, in the general case of arbitrary $\boldsymbol{v}, \boldsymbol{H}$ and ρ, the tangential disturbance with small perturbations will conserve its form, and boundary equations will assume the following form:

$$\left.\begin{aligned} v_n' &\equiv v_x' - \frac{\partial \xi}{\partial t} - v_y \frac{\partial \xi}{\partial y} - v_z \frac{\partial \xi}{\partial z} = 0; \\ h_n &\equiv h_x - H_y \frac{\partial \xi}{\partial y} - H_z \frac{\partial \xi}{\partial z} = 0; \\ &\left\{p' + \frac{1}{4\pi}(H_y h_y + H_z h_z)\right\} = 0. \end{aligned}\right\} \tag{73}$$

However, with some special values of the parameters of a nonperturbed disturbance of equation (72), there are possible some nontrivial solutions for h_n and v_n'. The values of these parameters characterize the transition disturbance, since even a small perturbation may cause a tangential disturbance to change into another type. Indeed, a disturbance where $h_n \neq 0$ or $v_n' \neq 0$ cannot be classified in the tangential type. Equations (72) permit the solution where $h_n = 0$, but where $v_n' \neq 0$, the original disturbance will satisfy the following conditions:

$$\left.\begin{aligned} \{v_y\} = 0; \quad \{v_z\} = 0; \quad \left\{\frac{H_y}{\rho}\right\} = 0; \quad \left\{\frac{H_z}{\rho}\right\} = 0; \\ \left\{w + \frac{H^2}{4\pi\rho}\right\} = 0. \end{aligned}\right\} \tag{74}$$

As shown in Section 3, the above discontinuity is thermodynamically possible.

If in the original tangential disturbance

$$\{\rho\} = 0, \tag{75}$$

then equations (72) permit h_n and v_n' to be different from zero if:

$$\{v_y\} = \pm \frac{1}{\sqrt{4\pi\rho}} \{H_y\}; \quad \{v_z\} = \pm \frac{1}{\sqrt{4\pi\rho}} \{H_z\}. \tag{76}$$

The above shows that a coordinate system may be selected in such a manner as to satisfy the following equation on both sides of the surface:

$$\boldsymbol{v} = \pm \frac{\boldsymbol{H}}{\sqrt{4\pi\rho}}. \tag{77}$$

From conditions (71) and (72) it follows that:

$$\left\{w + \frac{H^2}{4\pi\rho}\right\} = 0 \quad \text{and} \quad \{\varepsilon\} = 0. \tag{78}$$

Such a disturbance [refer to equation (60a)] is the transition between a tangential and magnetohydrodynamic wave. The perturbation of such a disturbance will satisfy the conditions:

$$\{v_n'\} = 0; \quad \{h_n\} = 0; \quad \left\{p' + \frac{1}{4\pi}(H_y h_y + H_z h_z)\right\} = 0 \tag{79}$$

and

$$v_n' = \pm \frac{h_n}{\sqrt{4\pi\rho}} \quad \text{and} \quad v_x' - \frac{\partial \xi}{\partial t} = \pm \frac{h_x}{\sqrt{4\pi\rho}}. \tag{80}$$

It follows from the above that even an arbitrarily small normal component of the field intensity will change the disturbance into a magnetohydrodynamic wave. Finally, if in the original tangential disturbance

$$\{\rho\} \neq 0, \tag{81}$$

then equations (72) will permit a nontrivial solution for h_n and v_n' in a case where the motion is planar, i.e.

$$v_z = 0, \quad H_z = 0 \tag{82}$$

and the following conditions exist:

$$\left.\begin{aligned} \{v_y\}^2 &= \frac{1}{4\pi}\left\{\frac{H_y}{\rho}\right\}\{H_y\}; \\ \left\{\frac{v^2}{2} + w + \frac{H^2}{4\pi\rho}\right\}\{H_y\} &= \{vH\}\,\{v_y\}. \end{aligned}\right\} \tag{83}$$

Indeed; with the selection of the coordinate system where $v_{z_1} = 0$ and $v_{z_2} = 0$, it will be found from (72) that with $h_n \neq 0$ and $v_n' \neq 0$:

$$\{H_z\} = 0; \ \left\{\frac{H_z}{\rho}\right\} = 0.$$

The above and condition (81) lead to equation (82). Excluding h_n and $\rho v_n'$ from the remaining equations (72), we get equations (83). The disturbance, determined by conditions (70) and (81) to (83) as shown in Section 3, is the transition between a tangential and an inclined shock wave. From equations (71) and (72) for small perturbations of such a disturbance:

$$\{\rho v_n'\} = 0, \quad \{h_n\} = 0, \ \left\{p' + \frac{1}{4\pi}H_y h_y\right\} = 0 \tag{84}$$

and

$$\rho v_n' = \frac{\{H_y\}}{4\pi\,\{v_y\}}\,h_n. \tag{85}$$

It is convenient to transform equations (83) to a system of coordinates where:

$$v_1 = \frac{H_1}{\rho_1}\,\frac{\{v\}}{\left\{\frac{H}{\rho}\right\}}, \quad v_2 = \frac{H_2}{\rho_2}\,\frac{\{v\}}{\left\{\frac{H}{\rho}\right\}}. \tag{86}$$

In such a system of coordinates, conditions (70) and (83) will assume the following form:

$$\left.\begin{aligned} &v_x = 0; \ H_x = 0; \qquad \{v_y\}^2 = \frac{1}{4\pi}\left\{\frac{H_y}{\rho}\right\}\{H_y\}; \\ &\left\{p + \frac{H^2}{8\pi}\right\} = 0; \qquad \left\{\frac{v_y^2}{2} + w\right\} = 0. \end{aligned}\right\} \tag{87}$$

It also follows from equations (85) and (86) that:

$$\frac{v_{n_1}'}{h_{n_1}} = \frac{v_{y_1}}{H_{y_1}} \ \text{and} \ \frac{v_{n_2}'}{h_{n_2}} = \frac{v_{y_2}}{H_{y_2}}, \tag{88}$$

i.e. the motion takes place along the force lines. This means that the discontinuity changes into a inclined shock wave, and the coordinate system (86) corresponds to the system of coordinates (43).

Thus, the investigation of perturbed limiting equations leads to the same results in relationship to transitions as direct comparison of limiting equations does for disturbances of different types.

It may be noted that the existence of transition disturbances does not mean that the motion is unstable in the sense of transfer of energy from the basic to the perturbed motion. According to expressions (80) and (85), only a small perturbance will change the disturbance from a transition form to another type.

SECTION 5. STABILITY OF A TANGENTIAL DISTURBANCE IN AN INCOMPRESSIBLE MEDIUM

In the case of an incompressible medium (ρ = constant), it is convenient to introduce the following quantity for the intensity of the magnetic field:

$$\boldsymbol{u} = \frac{\boldsymbol{H}}{\sqrt{4\pi\rho}}. \tag{89}$$

The system of magnetohydrodynamic equations (24) will then assume the following form:

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{u}}{\partial t} = (\boldsymbol{u}\nabla)\,\boldsymbol{v} - (\boldsymbol{v}\nabla)\,\boldsymbol{u}; \\ &\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\nabla)\,\boldsymbol{v} = -\frac{1}{\rho}\nabla\left(p + \frac{\rho u^2}{2}\right) + (\boldsymbol{u}\nabla)\,\boldsymbol{u}; \\ &\operatorname{div}\boldsymbol{v} = 0; \quad \operatorname{div}\boldsymbol{u} = 0. \end{aligned}\right\} \tag{90}$$

A solution corresponding to a tangential disturbance can be selected as the original steady state solution for the system of equations (90). The behavior of this solution during small perturbations is investigated. Let the motion be described by the following quantities:

$$\boldsymbol{v} + \boldsymbol{v}', \quad \boldsymbol{u} + \boldsymbol{u}', \quad p + p', \tag{91}$$

where $\boldsymbol{v}$, $\boldsymbol{u}$ and p (where indices 1 or 2 correspond to the investigated side of the surface) depend upon both the coordinates and time and satisfy conditions (32) to (34) for a tangential disturbance. In addition, $\boldsymbol{v}'$, $\boldsymbol{u}' = \frac{\boldsymbol{H}'}{\sqrt{4\pi\rho}}$ and p' are quantities which characterize a small perturbation. Substituting expressions (91) in equations (90) and, neglecting small quantities, the following system of equations will be found for the perturbations:

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{u}'}{\partial t} = (\boldsymbol{u}\nabla)\,\boldsymbol{v}' - (\boldsymbol{v}\nabla)\,\boldsymbol{u}'; \\ &\frac{\partial \boldsymbol{v}'}{\partial t} + (\boldsymbol{v}\nabla)\,\boldsymbol{v}' = -\frac{1}{\rho}\nabla(p' + \rho\boldsymbol{u}\boldsymbol{u}') + (\boldsymbol{u}\nabla)\,\boldsymbol{u}'; \\ &\operatorname{div}\boldsymbol{v}' = 0; \quad \operatorname{div}\boldsymbol{u}' = 0. \end{aligned}\right\} \tag{92}$$

The solution to the system of equations (92) on each side of the surface, where $x = 0$, is found by superposition of waves:

$$e^{i\,(\boldsymbol{k}\boldsymbol{r} - \omega t)}, \tag{93}$$

where $\boldsymbol{k} = (k_x, k_y, k_z)$. The components of vector

$$\boldsymbol{k}_0 = (0, k_y, k_z) \tag{94}$$

should be real, since otherwise the solution of equation (93) will reach infinity along the y or z axes. For perturbations that depend upon coordinates and time according to equation (93), equations (92) will be reduced to the following:

$$\left.\begin{aligned}(\omega - \boldsymbol{k}_0\boldsymbol{v})\,\boldsymbol{u}' &= -(\boldsymbol{k}_0\boldsymbol{u})\,\boldsymbol{v}';\\ (\omega - \boldsymbol{k}_0\boldsymbol{v})\,\boldsymbol{v}' &= \frac{1}{\rho}(p' + \rho\boldsymbol{u}\boldsymbol{u}')\,\boldsymbol{k} - (\boldsymbol{k}_0\boldsymbol{u})\,\boldsymbol{u}';\\ \boldsymbol{k}\boldsymbol{v}' = 0; &\quad \boldsymbol{k}\boldsymbol{u}' = 0.\end{aligned}\right\} \tag{95}$$

It is considered in the above that $v_x = 0$ and $H_x = 0$. Second equation multiplied by $\boldsymbol{k}$ leads to the condition:

$$k^2(p' + \rho\boldsymbol{u}\boldsymbol{u}') = 0. \tag{96}$$

It follows from the above that either $p' + \rho\boldsymbol{u}\boldsymbol{u}' = 0$, or $k^2 = 0$. In the first case, the following equation for ω may be derived from equations (95):

$$(\omega - \boldsymbol{k}_0\boldsymbol{v})^2 - (\boldsymbol{k}_0\boldsymbol{u})^2 = 0. \tag{97}$$

The roots for the above are real. Since the instability may be represented only by complex values of ω, the following will be assumed:

$$k^2 = 0; \quad k_x = \pm ik_0. \tag{98}$$

The sign of k_x is chosen from boundary conditions of perturbations removed from the disturbance; i.e. minus in the region 1 (with $x < 0$) and plus in region 2 (with $x > 0$). It follows from equations (95):

$$\boldsymbol{v}' = -\frac{\omega - \boldsymbol{k}_0\boldsymbol{v}}{\boldsymbol{k}_0\boldsymbol{u}}\,\boldsymbol{u}', \tag{99}$$

$$p' + \rho\boldsymbol{u}\boldsymbol{u}' = -\frac{\rho}{k_x}\,\frac{(\omega - \boldsymbol{k}_0\boldsymbol{v})^2 - (\boldsymbol{k}_0\boldsymbol{u})^2}{\boldsymbol{k}_0\boldsymbol{u}}\,u'_x. \tag{100}$$

Equations (99) and (100) are combined on the surface of a tangential disturbance for regions where $x < 0$ and $x > 0$, by means of boundary conditions (73). For solutions of the type of equation (93), these equations may be reduced to the following:

$$\left.\begin{aligned}u'_{x_1} - i(\boldsymbol{k}_0\boldsymbol{u}_1)\,\xi &= 0;\\ u'_{x_2} - i(\boldsymbol{k}_0\boldsymbol{u}_2)\,\xi &= 0;\\ \{p' + \rho\,\boldsymbol{u}\boldsymbol{u}'\} &= 0;\end{aligned}\right\} \tag{101}$$

when $x = 0$. The last equation demands that the quantities $\boldsymbol{k}_0$ and ω be continuous. The above and equation (100) lead to the following condition:

$$\rho_1\,\frac{(\omega - \boldsymbol{k}_0\boldsymbol{v}_1)^2 - (\boldsymbol{k}_0\boldsymbol{u}_1)^2}{\boldsymbol{k}_0\boldsymbol{u}_1}\,u'_{x_1} = -\rho_2\,\frac{(\omega - \boldsymbol{k}_0\boldsymbol{v}_2)^2 - (\boldsymbol{k}_0\boldsymbol{u}_2)^2}{\boldsymbol{k}_0\boldsymbol{u}_2}\,u'_{x_2}. \tag{102}$$

The condition that $k_{x_1} = -k_{x_2}$ is utilized here, and it is assumed that in a general case, densities ρ_1 and ρ_2 are different on both sides of the disturbance. Excluding quantities ξ, u'_{x_1} and u'_{x_2} , and from equations (101) and (102), an equation may be found which determines possible values of ω:

$$\rho_1(\omega - \boldsymbol{k}_0\boldsymbol{v}_1)^2 + \rho_2(\omega - \boldsymbol{k}_0\boldsymbol{v}_2)^2 - \rho_1(\boldsymbol{k}_0\boldsymbol{u}_1)^2 - \rho_2(\boldsymbol{k}_0\boldsymbol{u}_2)^2 = 0. \tag{103}$$

Hence

$$\begin{aligned}\omega = \frac{1}{\rho_1 + \rho_2}\{\rho_1(\boldsymbol{k}_0\boldsymbol{v}_1) + \rho_2(\boldsymbol{k}_0\boldsymbol{v}_2) \pm\\ \pm\sqrt{(\rho_1 + \rho_2)\,[\rho_1(\boldsymbol{k}_0\boldsymbol{u}_1)^2 + \rho_2(\boldsymbol{k}_0\boldsymbol{u}_2)^2] - \rho_1\rho_2\,[\boldsymbol{k}_0(\boldsymbol{v}_2 - \boldsymbol{v}_1)]^2}\}.\end{aligned} \tag{104}$$

Therefore, it follows that with the condition

$$\rho_1 (\boldsymbol{k}_0 \boldsymbol{u}_1)^2 + \rho_2 (\boldsymbol{k}_0 \boldsymbol{u}_2)^2 - \frac{\rho_1 \rho_2}{\rho_1 + \rho_2} [\boldsymbol{k}_0 (\boldsymbol{v}_2 - \boldsymbol{v}_1)]^2 < 0, \tag{105}$$

one of the roots of ω has a positive imaginary part, i e. there are solutions to equation (93) which are exponential with respect to time. This denotes the instability of the original tangential disturbance. In cases where condition (105) is not fulfilled, the original disturbance will be stable.

Denoting the step-change of velocity in the disturbance by:

$$\boldsymbol{v}_0 = \boldsymbol{v}_2 - \boldsymbol{v}_1 \tag{106}$$

and according to formula (89) it may be found that the tangential disturbance is stable in relation to the wave vector $\boldsymbol{k}_0$, if:

$$(\boldsymbol{k}_0 \boldsymbol{H}_1)^2 + (\boldsymbol{k}_0 \boldsymbol{H}_2)^2 - \frac{4\pi\rho_1\rho_2}{\rho_1 + \rho_2} (\boldsymbol{k}_0 \boldsymbol{v}_0)^2 \geqslant 0. \tag{107}$$

It is apparent from the above that the magnetic field will always introduce a positive correction to the left side of the inequality, i.e. it will exert a stabilizing action on the motion. Condition (107) does not depend upon the absolute value of the vector $\boldsymbol{k}_0$. With non-parallel values of $\boldsymbol{H}_1$, $\boldsymbol{H}_2$ and $\boldsymbol{v}_0$ it depends upon the direction of the vector $\boldsymbol{k}_0$. The condition of stability of a disturbance in all possible perturbations is determined by the minimum values of the left side of inequality (107). In cases where the y axis of the coordinate system is directed along the step-change of velocity, then:

$$\frac{(H_{y_1} H_{z_2} - H_{z_1} H_{y_2})^2}{4\pi (H_{z_1}^2 + H_{z_2}^2)} - \frac{\rho_1 \rho_2}{\rho_1 + \rho_2} v_0^2 \geqslant 0. \tag{108}$$

The above shows that the contribution of the magnetic field is always positive with the exception of the case where vectors $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ have a direction which does not correspond to the direction of the step-change of velocity. In this case, from condition (107), there exists a region of such directions of the wave vector $\boldsymbol{k}_0$, where the corresponding disturbances are unstable. The motion with non-parallel values of $\boldsymbol{H}$ and $\boldsymbol{v}$ is generally unstable, since a deformation of the magnetic field takes place. Of great interest is the case where the magnetic field is parallel to the step-change of velocity, i.e. where motion takes place along the field. In this case the motion may be stable in an arbitrary field as shown in Section 9. If $\boldsymbol{H}_1$, $\boldsymbol{H}_2$ and $\boldsymbol{v}_0$ are parallel, then condition (107) for all perturbations will assume the following form:

$$\frac{1}{4\pi} (H_1^2 + H_2^2) \geqslant \frac{\rho_1 \rho_2}{\rho_1 + \rho_2} v_0^2. \tag{109}$$

If, however, the magnetic field and density of the medium are the same on both sides of the surface, then condition (109) will reduce to the following:

$$\frac{H^2}{8\pi} \geqslant \frac{1}{4} \frac{\rho v_0^2}{2}. \tag{110}$$

Since $H^2/8\pi$ is the density of magnetic energy, and $\rho v_0^2/2$ is the density of kinetic energy, then condition (110) will mean that stability of a tangential disturbance along the magnetic field will take place when the magnetic energy of the medium becomes comparable with its kinetic energy.

An investigation analogous to the above shows that magnetohydrodynamic waves, in an incompressible medium, are always stable to small perturbations. In particular, the transition between a tangential disturbance and a magnetohydrodynamic wave will be stable. The latter may be proved if conditions (107) are combined with equations (75) and (77), which characterize such transitions. The condition of stability will then assume the following form:

$$(\boldsymbol{k}_0\boldsymbol{u}_1 + \boldsymbol{k}_0\boldsymbol{u}_2)^2 \geqslant 0 \tag{111}$$

and, of course, the above conditions will always be fulfilled. A small perturbation of a transient disturbance will remain small during the subsequent period.

SECTION 6. STABILITY OF TANGENTIAL DISTURBANCES IN A COMPRESSIBLE MEDIUM

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Using the same nomenclature as in previous sections and denoting by $\boldsymbol{u} = \frac{\boldsymbol{H}}{\sqrt{4\pi\rho}}$, $\boldsymbol{v}$, p, ρ the constant quantities, and also neglecting small quantities, the following equations will be derived from the system of equations (24):

$$\left.\begin{aligned}
&\frac{\partial \boldsymbol{u}'}{\partial t} + (\boldsymbol{v}\nabla)\,\boldsymbol{u}' = -\boldsymbol{u}\operatorname{div}\boldsymbol{v}' + (\boldsymbol{u}\nabla)\,\boldsymbol{v}';\\
&\frac{\partial \boldsymbol{v}'}{\partial t} + (\boldsymbol{v}\nabla)\,\boldsymbol{v}' = -\frac{1}{\rho}\nabla p' - \nabla(\boldsymbol{u}\boldsymbol{u}') + (\boldsymbol{u}\nabla)\,\boldsymbol{u}';\\
&\frac{\partial \rho'}{\partial t} + \boldsymbol{v}\nabla\rho' = -\rho\operatorname{div}\boldsymbol{v}';\\
&\operatorname{div}\boldsymbol{u}' = 0;\quad p' = c^2\rho'.
\end{aligned}\right\} \tag{112}$$

For perturbations, whose dependence upon coordinates and time has the form $e^{i(\boldsymbol{k}\boldsymbol{r} - \omega t)}$, this system may be reduced to the following algebraic equations:

$$\left.\begin{aligned}
&(\omega - \boldsymbol{k}\boldsymbol{v})\,\boldsymbol{u}' + (\boldsymbol{k}\boldsymbol{u})\,\boldsymbol{v}' - (\boldsymbol{k}\boldsymbol{v}')\,\boldsymbol{u} = 0;\\
&(\omega - \boldsymbol{k}\boldsymbol{v})\,\boldsymbol{v}' - (\boldsymbol{u}\boldsymbol{u}')\,\boldsymbol{k} + (\boldsymbol{k}\boldsymbol{u})\,\boldsymbol{u}' - \frac{c^2}{\rho}\,\rho'\boldsymbol{k} = 0;\\
&(\omega - \boldsymbol{k}\boldsymbol{v})\,\rho' - \rho\,(\boldsymbol{k}\boldsymbol{v}') = 0;\\
&(\boldsymbol{k}\boldsymbol{u}') = 0.
\end{aligned}\right\} \tag{113}$$

It is convenient for further consideration to denote:

$$\omega_0 = \omega - \boldsymbol{k}\boldsymbol{v} \tag{114}$$

where ω_0 is the frequency in the coordinate system in which the medium is stationary, and equations (113) can now be rewritten as:

$$\left.\begin{aligned}
&\omega_0\boldsymbol{u}' + (\boldsymbol{k}\boldsymbol{u})\,\boldsymbol{v}' - \omega_0\boldsymbol{u}\,\frac{\rho'}{\rho} = 0;\\
&(\boldsymbol{k}\boldsymbol{u})\,\boldsymbol{u}' + \omega_0\boldsymbol{v}' - \frac{\omega_0^2}{k^2}\,\boldsymbol{k}\,\frac{\rho'}{\rho} = 0;\\
&(\boldsymbol{u}\boldsymbol{u}') = \frac{\omega_0^2 - k^2c^2}{k^2}\,\frac{\rho'}{\rho};\\
&(\boldsymbol{k}\boldsymbol{u}') = 0.
\end{aligned}\right\} \tag{115}$$

It may be noted that due to the parallel form of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in the plane of the tangential disturbance where $x = 0$, the vector $\boldsymbol{k}_0 = (0, k_y, k_z)$ will enter into expressions $\boldsymbol{ku}$ and $\boldsymbol{kv}$ only with its real components. Neglecting the known steady state solutions with real values of ω, which satisfy the equation:

$$\omega_0^2 - (\boldsymbol{k}_0\boldsymbol{u})^2 = 0, \tag{116}$$

it is found from the first two equations of the system (115) that:

$$\boldsymbol{v}' = \frac{\omega_0}{\rho k^2} \frac{\omega_0^2 \boldsymbol{k} - k^2(\boldsymbol{k}_0\boldsymbol{u})\,\boldsymbol{u}}{\omega_0^2 - (\boldsymbol{k}_0\boldsymbol{u})^2} \rho'; \tag{117}$$

$$\boldsymbol{u}' = \frac{\omega_0^2}{\rho k^2} \frac{k^2\boldsymbol{u} - (\boldsymbol{k}_0\boldsymbol{u})\,\boldsymbol{k}}{\omega_0^2 - (\boldsymbol{k}_0\boldsymbol{u})^2} \rho'. \tag{118}$$

Substituting the above in the third equation of the system (115), the following condition of correspondence will be found:

$$\omega_0^4 - (c^2 + u^2)\,k^2\omega_0^2 + k^2c^2(\boldsymbol{k}_0\boldsymbol{u})^2 = 0. \tag{119}$$

The above determines ω_0 as a function of the wave vector $\boldsymbol{k}$:

$$\omega_0^2 = \frac{k^2}{2}\left[c^2 + u^2 \pm \sqrt{(c^2+u^2)^2 - 4c^2\frac{(\boldsymbol{k}_0\boldsymbol{u})^2}{k^2}}\right]. \tag{120}$$

It follows from equation (120) that real values of $\boldsymbol{k}$ correspond to the real values of k_x, since the values of k_y and k_z are always real. Therefore, the steady state harmonics will also correspond to the real value of ω_0. Thus, instability may be caused only by those perturbations for which $\operatorname{Im}(k_x) \neq 0$, and:

$$\lambda = ik_x. \tag{121}$$

The perturbation will be limited at a distance from the disturbance, if the following conditions are fulfilled:

$$\operatorname{Re}(\lambda_1) > 0; \quad \operatorname{Re}(\lambda_2) < 0. \tag{122}$$

Here λ_1 and λ_2 relate to the regions where $x < 0$ and $x > 0$ respectively. From equation (119) we get:

$$\lambda^2 = k_0^2 - \frac{\omega_0^4}{(c^2+u^2)\,\omega_0^2 - c^2(\boldsymbol{k}_0\boldsymbol{u})^2}. \tag{123}$$

Boundary equations (73) have to be fulfilled on the surface of a tangential disturbance. For perturbances of the investigated type, equation (73) will be reduced to:

$$\left.\begin{aligned} v_x' + i(\omega - \boldsymbol{k}_0\boldsymbol{v})\,\xi &= 0;\\ u_x' - i(\boldsymbol{k}_0\boldsymbol{u})\,\xi &= 0;\\ \{p' + \rho\,\boldsymbol{uu}'\} &= 0. \end{aligned}\right\} \tag{124}$$

The first two conditions, according to equations (117) and (118), correspond to each other, and therefore it is sufficient to limit the solution by the two last equations. Excluding ξ, the conditions may be written as:

$$\frac{u'_{x_1}}{k_0 u_1} = \frac{u'_{x_2}}{k_0 u_2}; \tag{125}$$

$$c_1^2 \rho_1' + \rho_1 u_1 u_1' = c_2^2 \rho_2' + \rho_2 u_2 u_2'. \tag{126}$$

Excluding by means of equation (118) the values of u' and ρ', it will be found that:

$$\frac{\lambda_1}{\rho_1 [\omega_{0_1}^2 - (k_0 u_1)^2]} = \frac{\lambda_2}{\rho_2 [\omega_{0_2}^2 - (k_0 u_2)^2]}. \tag{127}$$

Substituting equations (114) and (123) in the above, and assuming that conditions (122) apply, the following equation may be arrived at, which determines the possible values of ω:

$$\frac{\sqrt{k_0^2 - \dfrac{(\omega - k_0 v_1)^4}{(c_1^2 + u_1^2)(\omega - k_0 v_1)^2 - c_1^2 (k_0 u_1)^2}}}{\rho_1 [(\omega - k_0 v_1)^2 - (k_0 u_1)^2]} = - \frac{\sqrt{k_0^2 - \dfrac{(\omega - k_0 v_2)^4}{(c_2^2 + u_2^2)(\omega - k_0 v_2)^2 - c_2^2 (k_0 u_2)^2}}}{\rho_2 [(\omega - k_0 v_2)^2 - (k_0 u_2)^2]} \tag{128}$$

From the two possible values of each of the above radicals, the one which is positive should be selected. Thus, the investigation of the stability of a tangential disturbance in a compressible medium will be reduced to the investigation of roots of equation (128). For some values of the parameters

$$u_1;\ u_2;\ v_0 = v_2 - v_1;\ \rho_1;\ \rho_2;\ c_1;\ c_2, \tag{129}$$

equation (128) will not yield roots of ω having a positive real part for any value of k_0 The tangential disturbance, which is characterized by such values of parameters (129), will be stable in relationship to any perturbations. In a converse case, there will exist infinitely small perturbations that will result in instability.

In the limiting case where $c_1 \to \infty$ and $c_2 \to \infty$, which corresponds to an incompressible liquid, equation (128) may be reduced to equation (103). On the other hand, with $H_1 = 0$ and $H_2 = 0$, i.e. during the absence of a magnetic field, a problem of simple hydrodynamics, referring to the stability of tangential disturbances in a compressible medium (Ref. [18]), will result.

With arbitrary values of parameters (129), the general investigation of the roots of equation (128) is made difficult by the fact that in this equation, only strictly determined parts of the radicals must be selected. The solution of equation (128) leads to an algebraic expression containing tenth powers of ω. This operation, however, leads to the formation of additional roots that do not satisfy the original equation. Therefore, each squared root of the equation should be checked as to its pertinence to the original equation. Only the most interesting cases of disturbances will be dealt with here, i.e.., those disturbances where the motion of the medium takes place along a magnetic field and where the following vectors are parallel:

$$u_1,\ u_2,\ v_1 \text{ and } v_2. \tag{130}$$

By defining

$$\nu = \cos(\widehat{k_0, u}) \tag{131}$$

as the cosine of the angle between the direction of wave vector k_0 and the general direction of vectors (130), equation (128) may be written:

$$\frac{\sqrt{1-\dfrac{\left(\dfrac{\omega}{k_0}-v_1\nu\right)^4}{(c_1^2+u_1^2)\left(\dfrac{\omega}{k_0}-v_1\nu\right)^2-c_1^2u_1^2\nu^2}}}{\rho_1\left[\left(\dfrac{\omega}{k_0}-v_1\nu\right)^2-u_1^2\nu^2\right]}=$$
$$=-\frac{\sqrt{1-\dfrac{\left(\dfrac{\omega}{k_0}-v_2\nu\right)^4}{(c_2^2+u_2^2)\left(\dfrac{\omega}{k_0}-v_2\nu\right)^2-c_2^2u_2^2\nu^2}}}{\rho_2\left[\left(\dfrac{\omega}{k_0}-v_2\nu\right)^2-u_2^2\nu^2\right]}. \tag{132}$$

In the above, all real parts of the radicals are assumed to be positive. Investigation is made of the form of the imaginary roots of the above equation with $\nu \to 0$. Equation (132) with $\nu = 0$, despite the existence of a multiple root $\omega = 0$, contains only two simple real roots. Since in equations with real coefficients the transition from a real root to an imaginary root is possible only through a multiple root, then the imaginary roots of equation (132), if present, should go to zero together with ν. With small values of ν, this permits us to neglect the second term under the radical sign. Then ω will be determined by the following equation:

$$\rho_1\left[\left(\frac{\omega}{k_0}-v_1\nu\right)^2-u_1^2\nu^2\right]=-\rho_2\left[\left(\frac{\omega}{k_0}-v_2\nu\right)^2-u_2^2\nu^2\right]. \tag{133}$$

The roots of the above equation are either real or imaginary depending whether the following condition is fulfilled:

$$\rho_1u_1^2+\rho_2u_2^2-\frac{\rho_1\rho_2}{\rho_1+\rho_2}(v_2-v_1)^2 \gtrless 0. \tag{134}$$

The above condition corresponds to condition (109), which was arrived at for an analogous problem in an incompressible liquid. Thus, with small values of ν, i.e. for perturbations for which the wave vector makes a large angle with the direction of the step-change of velocity, the compressibility of the medium is not important. The above result is physically apparent. Indeed, the behavior of perturbation depends only upon the projection of the step-change of velocity upon the direction of the wave vector of perturbation, and not upon the absolute value of the step-change of velocity. With sufficiently large angles, the above projection is quite small in comparison with the velocity of sound, so that the medium behaves in an incompressible manner.

The above result permits us to conclude that for disturbances (130) the compressibility of the medium does not lead to contraction of the region of instability in comparison with an incompressible medium. In cases where some values of the parameters cause an instability of the tangential disturbance in an incompressible medium, then in an analogous situation in a compressible medium an unstable condition will also prevail. However, in an incompressible medium, the non-fulfillment of condition (134) will lead to instability of any perturbation; whereas in a compressible medium the disturbance appears to be stable in relation to some parts of the perturbations (not with small values of ν). The above does not generally produce instability of the disturbances.

The case of arbitrary values of ν will now be investigated. The compressibility of the medium is of importance here. For disturbances in which the density of the medium, velocity of sound, and intensity of the field on both sides are equal:

$$\rho_1 = \rho_2 \equiv \rho, \qquad c_1 = c_2 \equiv c, \qquad u_1 = u_2 \equiv u. \tag{135}$$

It is convenient to utilize a "symmetrical" system of coordinates, where

$$v_1 = -\frac{v_0}{2}, \qquad v_2 = \frac{v_0}{2} \tag{136}$$

and v_0 is the magnitude of the step-change of velocity in the disturbance. Denoting

$$w = \frac{\omega}{k_0 c}, \qquad \alpha = \frac{u}{c} \text{ and } \beta = \frac{v_0}{2c}, \tag{137}$$

equation (132) may be rewritten as:

$$\frac{\sqrt{1 - \frac{(w+\beta\nu)^4}{(1+\alpha^2)(w+\beta\nu)^2 - \alpha^2\nu^2}}}{(w+\beta\nu)^2 - \alpha^2\nu^2} = -\frac{\sqrt{1 - \frac{(w-\beta\nu)^4}{(1+\alpha^2)(w-\beta\nu)^2 - \alpha^2\nu^2}}}{(w-\beta\nu)^2 - \alpha^2\nu^2}. \tag{138}$$

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It follows from conditions (133) that when $\nu \to 0$ the stable condition will assume the form of $\frac{H^2}{4\pi} \geqslant \frac{\rho v_0^2}{4}$ or, in terms of equation (137),

$$\alpha \geqslant \beta. \tag{139}$$

The region of stability can now be determined by the use of equation (138) when $\nu = 1$. This corresponds to "lengthwise" perturbations, which are propagated along the step-change in velocity. Equation (138) can now be reduced to the simple form:

$$w^4 - 2(1+\beta^2)w^2 + \beta^4 - 2\beta^2 + \frac{2\alpha^2}{1+\alpha^2} = 0. \tag{140}$$

The multiplier, which corresponds to the following real roots, was omitted:

$$w = 0 \text{ and } w = \pm(\alpha \pm \beta). \tag{141}$$

The roots of equation (140) are

$$w = \pm\sqrt{1 + \beta^2 \pm \sqrt{4\beta^2 + \frac{1-\alpha^2}{1+\alpha^2}}}, \tag{142}$$

whence it follows that imaginary roots are possible for two regions of parameters α and β as shown in Figure 7.

$$\text{a)} \qquad \frac{1-\alpha^2}{1+\alpha^2} > (1-\beta^2)^2. \tag{143}$$

This region corresponds to purely imaginary values of w, and the transition from real to imaginary values of the latter, with a continuous change of α and β goes through the value of $w = 0$.

$$\text{b)} \qquad \frac{\alpha^2 - 1}{\alpha^2 + 1} > 4\beta^2. \tag{144}$$

In this region, the real part of w differs from zero, and the change from its real to imaginary values goes through a multiple root that is not equal to zero.

The condition that determines region "a" in the limiting cases where $c \to \infty$ and $H = 0$ leads to known conditions of instability for an incompressible medium: $\frac{H^2}{2\pi} < \frac{\rho v^2}{2}$; while for a compressible medium without a magnetic field: $v_0 < 2\sqrt{2}C$. Thus, condition (143) represents a general case for a compressible medium in a magnetic field.

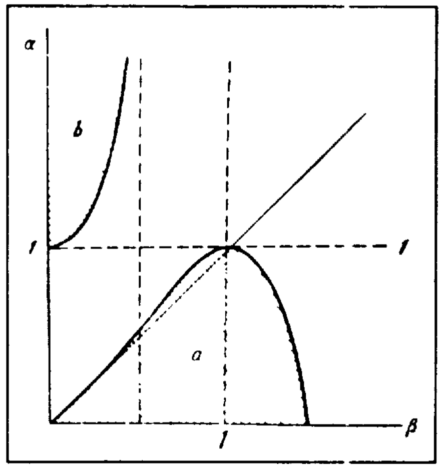

Fig. 7. Regions of imaginary roots of equation (140) (shaded areas).

Condition "b", however, leads to an unexpected conclusion, namely, that in very strong fields even a small disturbance in velocity causes instability. Moreover, squaring of both sides of the equation for w leads to the appearance of additional roots which do not satisfy the original equation. These roots correspond to unlimited solutions with $x = \pm\infty$ and should be neglected. The investigation of the roots of equation (142) in region "b" shows that such roots are not necessary, since they do not satisfy equation (138). Thus, the region of instability of the disturbances for longitudinal perturbations is determined by condition (143).

Two limiting cases are considered: $\nu \to 0$ and $\nu = 1$. In both cases the boundary of the unstable region corresponds to zero values of w. In this case, the symmetrical system of coordinates is employed, where both positive and negative roots of w are present, as follows directly from equation (138). During the transition from the stable to the unstable region, the value of w^2 changes from positive, through zero on the boundary, to negative in the region of instability. Here the values of w are either real or purely imaginary. Thus, the boundaries of stability are denoted by curves for which the value of w^2 changes its sign.

The change from real to imaginary values could take place through a multiple root which is other than zero, since in the unstable region both the real and imaginary parts of $w = w_r + iw_i$ would be different from zero. Such a condition would denote an instability of oscillations with growing amplitude, the dependence of which on time would be:

$$e^{\omega_i t} \cos \omega_r t.$$

However, in all investigated cases, instability of this type appears to be impossible. (Instability of this type is sometimes called "super-stability".) It occurs in the problem on thermal convection as described in Refs. [19] and [20]. It will be further assumed that with intermediate values of ν, between $\nu = 0$ and $\nu = 1$, such instability does not appear. Thus, it will be assumed that with intermediate values of ν, the boundary between regions of stability and instability (α, β) may be only such curves for which w^2 changes its sign. Equation (140) can be reduced to the following algebraic equation:

$$c_0 w^8 + c_1 w^6 + c_2 w^4 + c_3 w^2 + c_4 = 0, \tag{145}$$

where coefficients c are algebraic functions of the parameters α, β and ν. In particular, the constant c_4 is:

$$\begin{aligned} c_4 = (\alpha^2 - \beta^2)\{[2(1-\beta^2)^2 + \nu^2(2-\beta^2)\beta^2]\alpha^4 - \\ - \beta^2[4(1-\beta^2) + \nu^2\beta^2(1+\beta^2)]\alpha^2 + 2\beta^4 - \nu^2\beta^6\}. \end{aligned} \tag{146}$$

In cases where the constant c_4 becomes zero, then equation (145) will contain a root $w^2 = 0$. In cases where the constant changes its sign by transition through the value of zero, then the value of w^2 will also change its sign, and consequently there will exist pairs of real roots which will be transformed into a pair of complex and purely imaginary roots. In the latter case, when the constant equals zero with a fixed value of ν, a curve will be determined which will form the boundary of the region of stability. The maximum region in the plane (α, β) bounded by such curves represents the region of instability of the original tangential disturbance. This region will be investigated.

Equating expression (146) to zero, we get:

$$\alpha^2 - \beta^2 = 0 \tag{147a}$$

and

$$\begin{gathered}[2(1-\beta^2)^2 + \nu^2(2-\beta^2)\beta^2]\,\alpha^4 - \\ -\beta^2[4(1-\beta^2) + \nu^2\beta^2(1+\beta^2)]\,\alpha^2 + 2\beta^4 - \nu^2\beta^6 = 0.\end{gathered} \tag{147b}$$

Equation (147b) has the following roots:

$$\alpha^2 = \frac{4(1-\beta^2) + \nu^2\beta^2(1+\beta^2) \pm \nu\beta^2\sqrt{\nu^2[8+(1-\beta^2)^2] - 8}}{2[2(1-\beta^2)^2 + \nu^2\beta^2(2-\beta^2)]}\,\beta^2. \tag{148}$$

Equation (148) determines the real curve only for the case when

$$\nu^2 \geqslant \frac{8}{8+(1-\beta^2)^2}. \tag{149}$$

The above condition determines, for each value of ν, two intervals of the values of β determined by expression (148):

$$\beta^2 \leqslant 1 - 2\sqrt{2\left(\frac{1}{\nu^2} - 1\right)} \tag{150}$$

and

$$\beta^2 \geqslant 1 + 2\sqrt{2\left(\frac{1}{\nu^2} - 1\right)}. \tag{151}$$

Interval (150) permits the existence of real values of β only when the condition $8/9 \leqslant \nu^2 \leqslant 1$ exists. In the region where $\beta^2 \leqslant 1$, and with $\nu^2 < 8/9$, there is only one line (147a) that serves as the boundary of the stability region. Interval (151) with $\nu \to 0$ leads to infinity. In this case, with all finite values of $\beta \geqslant 1$ the boundary of the stability region will also be determined by line (147a). Thus, with small values of ν, the previously determined condition (139) may be used.

However, the $\nu = 1$ curve, determined by equation (148), merges with line (139) and curve (143). Indeed, with $\nu = 1$, the constant of (146) will assume the following form:

$$(\alpha^2 - \beta^2)^2\,[(1+\alpha^2)(1-\beta^2)^2 - (1-\alpha^2)]. \tag{152}$$

Therefore, on the straight line $\alpha = \beta$, the change of w^2 to zero will take place without changing of signs, and the following curve will form the boundary for the stability region:

$$(1+\alpha^2)(1-\beta^2)^2 - (1-\alpha^2) = 0, \tag{153}$$

which is in concurrence with condition (32).

Thus, with the change of ν from zero to plus one, a continuous deformation of the boundary of the stability region takes place as determined by equation (148), between two end conditions (139) and (153). For all values of ν, curves (148) are limited on the left side $\beta \geqslant 1$ by line (139), and with $\beta \leqslant 1$, by curve (153). Therefore, the complete region of instability is also limited by the same equations. Those disturbances whose parameters are outside of this region are stable in relationship to even infinitely small perturbations.

The boundary of the stable region of tangential disturbances of velocity in a compressible medium is shown on plane (α,β) of Figure 8. It is evident that the boundary of the stability region differs only slightly from the line $\alpha = \beta$. According to equations (137), $\frac{H^2}{8\pi} = \frac{1}{4}\frac{\rho v^2}{2}$, and the criterion of stability of a tangential disturbance of velocity in a compressible medium remains quantitatively the same as for an incompressible medium in accordance with equation (110). That is the disturbance is stable if the magnetic energy flux is comparable to the kinetic energy flux in relationship to the motion of the medium.

Therefore, in compressible as well as incompressible conducting media, a longitudinal magnetic field stabilizes the motion if its energy is comparable to the kinetic energy of this motion.

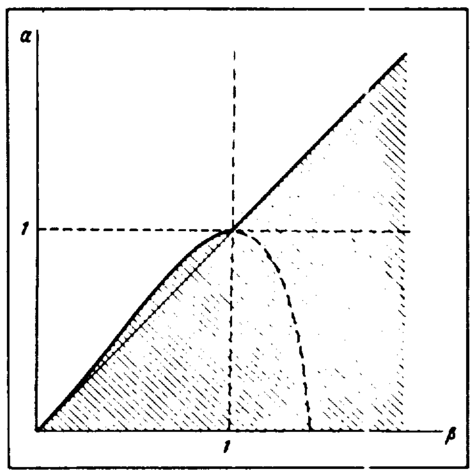

Fig. 8. Region of instability of tangential disturbances of velocity in compressible medium (shaded areas).

SECTION 7. THE STRUCTURE OF DISTURBANCES

Until now disturbances were considered as mathematical surfaces upon which the parameters that characterize motion of the medium undergo a disruption of continuity. In reality, the viscosity, thermal conductivity, and limited conductivity of the medium cause the surface to represent a region of rapid but continuous change of these parameters. The thickness of such a region is called the width of the disturbance. For determination of this width in equations of motion, the dissipation terms have to be taken into account. These terms have been neglected until now.

An investigation of a planar steady state disturbance that is parallel to the plane $x = 0$ and is homogeneous along the coordinates y and z, $\left(\frac{\partial}{\partial y} = 0, \frac{\partial}{\partial z} = 0\right)$ shall now be made. Magnetohydrodynamic equations (14), (17), and (22) for one-dimensional steady state motion will assume the following form by integration with respect to x:

$$\left.\begin{aligned}
&v_x H_y - v_y H_x - \beta \frac{dH_y}{dx} = \text{const};\\
&v_z H_x - v_x H_z + \beta \frac{dH_z}{dx} = \text{const};\\
&H_x = \text{const}; \qquad \rho v_x = \text{const};\\
&p + \rho v_x^2 + \frac{H^2}{8\pi} - \left(\frac{4}{3}\eta + \zeta\right)\frac{dv_x}{dx} = \text{const};\\
&\rho v_x v_y - \frac{1}{4\pi} H_x H_y - \eta \frac{dv_y}{dx} = \text{const};\\
&\rho v_x v_z - \frac{1}{4\pi} H_x H_z - \eta \frac{dv_z}{dx} = \text{const};\\
&\rho v_x\left(\frac{v^2}{2} + w + \frac{H^2}{4\pi\rho}\right) - \frac{1}{4\pi}(\boldsymbol{vH}) H_x - \frac{\beta}{8\pi}\frac{dH^2}{dx} -\\
&-\frac{1}{2}\left(\frac{4}{3}\eta + \zeta\right)\frac{dv_x^2}{dx} - \frac{\eta}{2}\frac{d}{dx}(v_y^2 + v_z^2) - \varkappa \frac{dT}{dx} = \text{const}.
\end{aligned}\right\} \tag{154}$$

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The above equations describe the motion of the medium in the disturbance. The integration of the right hand side of equations (154) is determined from the following conditions: all quantities far from the disturbance and on either side of it do not depend on x $\left(\frac{\partial}{\partial x} = 0\right)$, and are known on side 1 $(x < 0)$ and on side 2 $(x > 0)$. Also in accordance with equations (154), the quantities should be determined by relationships which correspond to general boundary equations (131) on the surface of the disturbance.

For a tangential disturbance $(v_x = 0,\ H_x = 0)$, equations (154) determine the following:

$$\eta \frac{d\boldsymbol{v}}{dx} = 0, \qquad \beta \frac{d\boldsymbol{H}}{dx} = 0 \tag{155}$$

(since all quantities far from the disturbance do not depend upon x, it is assumed that the constant is equal to zero). These conditions denote that the viscosity $(\eta \neq 0)$ and the limited conductivity of the medium $(\beta \neq 0)$ prevent the occurrence of a steady state tangential disturbance of velocity and of magnetic field: thus, the disturbance disappears as time goes on. The disappearance of the disturbance is determined by simple diffusion equations. Therefore, the relationship of the velocity of disappearance of the disturbance δ/τ (where δ is the width of the disturbance and τ is the duration of its existence) to the characteristic velocity of the flow V is inversely proportional to the Reynolds number R:

$$\frac{\delta}{\tau V} \approx \frac{\nu}{\delta V} = \frac{1}{\mathrm{R}},$$

in cases where the disappearance of a tangential disturbance of velocity occurs as a result of the action of viscosity $\nu = \eta/\rho$; or the Reynolds number R_m:

$$\frac{\delta}{\tau V} \approx \frac{\beta}{\delta V} = \frac{1}{\mathrm{R}_m},$$

in cases where the disappearance of a disturbance of magnetic field results from the limited conductivity of the medium.

As shown, the numbers R and R_m are very large in the majority of astrophysical applications and thus disappearance of a tangential disturbance may be neglected. Although a steady state tangential disturbance is not possible, the above concept may be widely used in astrophysics.

For the case of inclined shock waves and parallel shock waves(v_y, v_z, H_y, H_z are each zero), equations (154) are reduced to ordinary hydrodynamic equations of one-dimensional steady state motion. In this case, the magnetic field does not exert any influence upon the motion of the medium. In particular, the width of a shock wave of low intensity is determined by expression (21):

$$\delta = \frac{A}{p_2 - p_1}, \qquad A = \frac{4V^3}{c^3 \left(\frac{\partial^2 V}{\partial p^2}\right)_S} \left[\frac{4}{3}\eta + \zeta + \varkappa\left(\frac{1}{c_v} - \frac{1}{c_p}\right)\right], \tag{156}$$

where $V = 1/\rho$ is the specific volume, and c_v and c_p are the specific heats of the medium. Thus, considering dissipation terms in the equations of motion, the motion along the field takes place in the same manner as in ordinary hydrodynamics. In all remaining cases, the character of motion (including the width of the disturbance) actually depends upon the intensity of the magnetic field. The width of a perpendicular shock wave of low intensity is taken as an example. The perpendicular shock wave can be conveniently investigated in a system of coordinates where the quantities $v_x = v$ and $H_y = H$ differ from zero. The system of equations (154) can now be reduced to the following equations:

$$\left.\begin{aligned} & vH - \beta\frac{dH}{dx} = v_1 H_1; \\ & \rho v = j = \rho_1 v_1; \\ & p + \rho v^2 + \frac{H^2}{8\pi} - \left(\frac{4}{3}\eta + \zeta\right)\frac{dv}{dx} = p_1 + \rho_1 v_1^2 + \frac{H_1^2}{8\pi}; \\ & \rho v\left(\frac{v^2}{2} + w\right) + \frac{H^2}{4\pi}v - \left(\frac{4}{3}\eta + \zeta\right)v\frac{dv}{dx} - \frac{\beta}{8\pi}\frac{dH^2}{dx} - \\ & \quad - \varkappa\frac{dT}{dx} = \rho_1 v_1\left(\frac{v_1^2}{2} + w_1\right) + \frac{H_1^2}{4\pi}v_1 \end{aligned}\right\} \tag{157}$$

Here, j is the flow flux of matter, and subscript 1 denotes quantities that are removed from the disturbance in the region where $x < 0$. Furthermore, it is convenient to introduce the specific volume $V = 1/\rho$, and for the intensity of the magnetic field, the quantity

$$B = \frac{H}{\rho\sqrt{4\pi}} = \frac{HV}{\sqrt{4\pi}}. \tag{158}$$

Equations (157) will assume the following form:

$$v = jV; \tag{159}$$

$$B - B_1 = \frac{\beta}{jV}\frac{dB}{dx} - \frac{\beta B}{jV^2}\frac{dV}{dx} \tag{160}$$

$$p - p_1 + j^2(V - V_1) - \left(\frac{4}{3}\eta + \zeta\right)j\frac{dV}{dx} + \frac{1}{2}\left(\frac{B^2}{V^2} - \frac{B_1^2}{V_1^2}\right) = 0; \tag{161}$$

$$\begin{aligned} & w - w_1 + \frac{j^2}{2}(V^2 - V_1^2) + \frac{B^2}{V} - \frac{B_1^2}{V_1} - \left(\frac{4}{3}\eta + \zeta\right)jV\frac{dV}{dx} - \\ & \quad - \frac{\beta}{2j}\frac{d}{dx}\left(\frac{B^2}{V^2}\right) - \frac{\varkappa}{j}\frac{dT}{dx} = 0 \end{aligned} \tag{162}$$

Since disturbances of low intensity are investigated here, the differences between the external and internal quantities of the disturbance, $p - p_1$, $V - V_1$, $B - B_1$, etc., are small and only up to second order terms should be taken into account. It will be assumed that $1/\delta$ is a quantity of the same order of magnitude as $p - p_1$. Therefore, the differentiation of x changes the order of magnitude to unity. With the above assumptions, the first term on the right hand side in equations (160) may be neglected. Substituting B_1 for B in the second term, we get:

$$B - B_1 = -\frac{\beta B_1}{jV_1^2}\frac{dV}{dx}. \tag{163}$$

The quantity $H/\rho = \sqrt{4\pi}\,B$, expresses the "attachment" of the magnetic field to the medium and persists during the passage of a perpendicular shock wave. Its change has a second order of magnitude and reaches maximum inside of the disturbance.

Terms containing B in equations (161) and (162) are easily brought to yield:

$$\frac{1}{2}\left(\frac{B^2}{V^2} - \frac{B_1^2}{V_1^2}\right) = -\frac{B_1^2}{V_1^3}(V - V_1) - \frac{\beta B_1^2}{jV_1^4}\frac{dV}{dx} + \frac{3B_1^2}{2V_1^4}(V - V_1)^2; \tag{164}$$

$$\frac{B^2}{V} - \frac{B_1^2}{V_1} = -\frac{B_1^2}{V_1^2}(V - V_1) + \frac{B_1^2}{V_1^3}(V - V_1)^2 - \frac{2\beta B_1^2}{jV_1^3}\frac{dV}{dx}; \tag{165}$$

$$\frac{d}{dx}\left(\frac{B^2}{V^2}\right) = -\frac{2B_1^2}{V_1^3}\frac{dV}{dx}. \tag{166}$$

Further calculations are analogous to calculations known from ordinary hydrodynamics as shown in Ref. [21]. Multiplication of equation (161) by $1/2\,(V + V_1)$ and use of relationships (164) to (166) will show that terms containing B will cancel and

$$w - w_1 - \frac{1}{2}(V + V_1)(p - p_1) - \frac{\varkappa}{j}\frac{dT}{dx} = 0. \tag{167}$$

The magnetic field does not contribute to the above expression. It is seen from equation (167) that for terms of the lowest order of $p - p_1$ and $S - S_1$:

$$T(S - S_1) = \frac{\varkappa}{j}\left(\frac{\partial T}{\partial p}\right)_S \frac{dP}{dx}. \tag{168}$$

This means that the change of entropy inside the disturbance is a second-order-magnitude quantity in comparison with the change of pressure, and thus does not depend upon the magnetic field. Substituting equation (164) in equation (161) yields:

$$\begin{aligned} p - p_1 + \left(j^2 - \frac{B_1^2}{V_1^3}\right)(V - V_1) + \frac{3}{2}\frac{B_1^2}{V_1^4}(V - V_1)^2 &= \\ = \left[\left(\frac{4}{3}\eta + \zeta\right)j + \frac{\beta B_1^2}{jV_1^4}\right]\frac{dV}{dx} & \end{aligned} \tag{169}$$

or, substituting in the above the quantity V and using equation (168), the following may be arrived at:

$$\begin{aligned} &\frac{1}{2}\left[\left(j^2 - \frac{B_1^2}{V_1^3}\right)\left(\frac{\partial^2 V}{\partial p^2}\right)_S + 3\frac{B_1^2}{V_1^4}\left(\frac{\partial V}{\partial p}\right)_S^2\right](p - p_1)^2 + \\ &+ \left[1 + \left(j^2 - \frac{B_1^2}{V_1^3}\right)\left(\frac{\partial V}{\partial p}\right)_S\right](p - p_1) = \left\{\left[\left(\frac{4}{3}\eta + \zeta\right)j + \frac{\beta B_1^2}{jV_1^4}\right]\left(\frac{\partial V}{\partial p}\right)_S - \right. \\ &\left. - \frac{\varkappa}{jT}\left(j^2 - \frac{B_1^2}{V_1^3}\right)\left(\frac{\partial V}{\partial S}\right)_p\left(\frac{\partial T}{\partial p}\right)_S\right\}\frac{dp}{dx}. \end{aligned} \tag{170}$$

The left side of this equation may be transformed to become equivalent to the following equation:

$$\frac{1}{2}\left[\left(j^2-\frac{B_1^2}{V_1^3}\right)\left(\frac{\partial^2 V}{\partial p^2}\right)_S+3\frac{B_1^2}{V_1^4}\left(\frac{\partial V}{\partial p}\right)_S^2\right](p-p_1)(p-p_2)=$$
$$=\left\{\left[\left(\frac{4}{3}\eta+\zeta\right)j+\frac{\beta B_1^2}{jV_1^4}\right]\left(\frac{\partial V}{\partial p}\right)_S-\frac{\varkappa}{jT}\left(j^2-\frac{B_1^2}{V_1^3}\right)\left(\frac{\partial V}{\partial S}\right)_p\left(\frac{\partial T}{\partial p}\right)_S\right\}\frac{dp}{dx}. \tag{171}$$

Here, j is a function of shock-wave intensity p_2-p_1. Since terms only up to the second order of magnitude of $p-p_1$ are taken into account, then for $j=\rho v$ a null approximation may be used. In such an approximation, $\rho=\rho_1=\rho_2$, and the velocity of propagation of disturbance v, in relation to a quiescent medium, is equal to the velocity of propagation of a small perturbation whose direction is perpendicular to the magnetic field. The latter, according to equation (120), is equal to:

$$v=\frac{\omega_0}{k}=\sqrt{c^2+u^2}, \tag{172}$$

where $u=H/\sqrt{4\pi\rho}$. Therefore, in a null approximation,

$$j=\rho\sqrt{c^2+u^2}. \tag{173}$$

Taking into account the equations

$$B^2=\frac{H^2}{4\pi\rho^2}=\frac{u^2}{\rho},\qquad \left(\frac{\partial V}{\partial p}\right)_S=-\frac{1}{\rho^2c^2}$$

and

$$\left(\frac{\partial V}{\partial S}\right)_p\left(\frac{\partial T}{\partial p}\right)_S=T\left(\frac{1}{c_v}-\frac{1}{c_p}\right)\frac{1}{\rho^2c^2},$$

equation (171) can be transformed into the following:

$$\frac{dp}{dx}=-\frac{2}{A}(p-p_1)(p-p_2), \tag{174}$$

where

$$A=\frac{4c\left[\left(\frac{4}{3}\eta+\zeta\right)\left(1+\frac{u^2}{c^2}\right)+\rho\beta\frac{u^2}{c^2}+\varkappa\left(\frac{1}{c_v}-\frac{1}{c_p}\right)\right]}{\sqrt{1+\frac{u^2}{c^2}}\left[\rho^3c^4\left(\frac{\partial^2V}{\partial p^2}\right)_S+3\frac{u^2}{c^2}\right]}. \tag{175}$$

Integration of equation (174) shows the dependence of pressure in the disturbance upon the coordinate x:

$$p=\frac{p_1+p_2}{2}+\frac{p_2-p_1}{2}\,\mathrm{th}\left(\frac{p_2-p_1}{A}x\right). \tag{176}$$

In the above, x is assumed to start in the "middle" of the disturbance, i.e., from the plane where $p=\frac{p_1+p_2}{2}$. The latter shows that away from the disturbance, the pressure reaches asymptotically the value of p_1 when $x<0$, and the values of p_2 with $x>0$. The change of pressure takes place in a layer whose width is:

$$\delta=\frac{A}{p_2-p_1}. \tag{177}$$

Therefore, δ is the effective width of a perpendicular shock wave in a case when its intensity is small. It will be noted that $1/\delta$ has the order of magnitude of quantities $p - p_1$, as was assumed above. With absence of magnetic field, expressions (175) and (177) are transformed into simple expressions (156).

The result obtained is analogous to ordinary hydrodynamics with only this difference, that the width of the disturbance does not depend only upon viscosity and thermal conductivity of the medium but also upon its electrical conductivity. Moreover, the width of a strong disturbance does not necessarily have the order of magnitude of the average path of free flow of the particles of the medium. With small conductivity, i.e., with large values of β, the width of the disturbance may be significantly higher.

It should be noted, that conditions leading to the formation of a shock wave are not investigated here. In cases where conductivity of the medium is so small that the condition $\frac{l\sigma}{c} \gg 1$ is not fulfilled, simple shock waves may be created in the medium in accordance with equation (9). The interaction between simple shock waves and the magnetic field is weak and their widths are equal in order of magnitude to the quantity l.

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SECTION 8. PROPAGATION OF PERTURBATIONS IN STEADY STATE FLOW

Equations of magnetohydrodynamics contain solutions of the elliptical type, which describes the entire space, and the hyperbolic type corresponding to both the incoming and outgoing waves. In ordinary hydrodynamics the character of the solution is determined by the number $M = v/c$. Flows with subsonic velocities have an elliptical character, and flows with supersonic velocities are related to the hyperbolic type. There are two numbers in magnetohydrodynamics which determine the type of solution and consequently some regions of their values for which the flow is classified as the hyperbolic type.

An investigation is made of small perturbations of uniform steady state flow. Such perturbations are described by the system of equations (112), which can be reduced to the following equation:

$$\left[\left(\frac{D}{Dt}\right)^4 - (c^2 + u^2)\left(\frac{D}{Dt}\right)^2 \nabla^2 + c^2 (\boldsymbol{u}\nabla)^2 \nabla^2\right]\rho' = 0. \tag{178}$$

Here, $D/Dt = \partial/\partial t + (\boldsymbol{v}\nabla)$, and ∇ is an ordinary operator. It is considered that the nonperturbed flow takes place along the magnetic field, i.e., $\boldsymbol{v} \parallel H$, the direction of which is selected along the y axis. Also, considering that the disturbance is both two-dimensional $(\partial/\partial z = 0)$ and steady state $(\partial/\partial t = 0)$, equation (178) is reduced as follows:

$$\left(\frac{\partial^2}{\partial x^2} + \varkappa^2 \frac{\partial^2}{\partial y^2}\right)\frac{\partial^2 \rho'}{\partial y^2} = 0, \tag{179}$$

where

$$\varkappa^2 = \frac{(c^2 - u^2)(u^2 - v^2)}{c^2u^2 - c^2v^2 - u^2v^2}. \tag{180}$$

Introducing dimensionless numbers

$$M = \frac{v}{c} \text{ and } \alpha = \frac{u}{c}, \tag{181}$$

the equation for $\varkappa^2$ may be rewritten as:

$$\varkappa^2 = \frac{(1 - M^2)(\alpha^2 - M^2)}{\alpha^2 - M^2(1 + \alpha^2)}. \tag{182}$$

The type of motion is therefore determined by the sign of the coefficient $\varkappa^2$. Thus, the equation may be classified as the elliptical type where $\varkappa^2 > 0$, and as the hyperbolic type in the opposite case. It is noted that elliptical solutions correspond to the following regions of parameters α and M as shown on Figure 9:

$$\text{a) } M^2 < \frac{\alpha^2}{1+\alpha^2}; \quad \text{b) } M^2 < 1; \quad M^2 > \alpha^2; \quad \text{c) } 1 < M^2 < \alpha^2. \tag{183}$$

With the remaining values of numbers M and α, the equation relates to the hyperbolic type. Let us consider the problem of flow along a hard "wavy" wall, given by the equation:

$$x = \xi(y),$$

where

$$\xi(y) = \mathrm{Re}\,(\xi_0 e^{iky}). \tag{184}$$

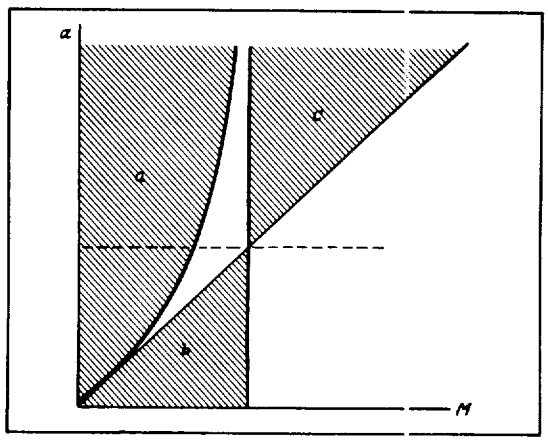

Fig. 9. Shaded areas denote the regions of M and α, where $\varkappa^2$ is positive.

It will be considered that the "waviness" is weak, i.e., $\xi_0 k \ll 1$. Here the flow differs little from uniform flow, and linear equations (112) or (178) may be employed. It may also be noted, that any non-uniformity on the plane boundary of the surface may be represented as superposition of the harmonics of equation (184). Therefore, the problem of flow around an arbitrary small obstacle reduces to the problem on flow around a "wavy" wall. For solutions, which depend on y in the form of e^{iky}, equation (179) reduces to the following:

$$\left(\frac{d^2}{dx^2} - k^2\varkappa^2\right)\rho' = 0 \tag{185}$$

and will yield the following general solution:

$$\rho' = (C_1 e^{k\varkappa x} + C_2 e^{-k\varkappa x})\, e^{iky}. \tag{186}$$

The remaining quantities are expressed in terms of ρ' in accordance with equations (117) to (119):

$$\left.\begin{aligned}
v'_y &= -\frac{c^2}{\rho v}\rho'; \\
u'_y &= \frac{u(v^2 - c^2)}{\rho v^2}\rho'; \\
v'_x &= \frac{u^2v^2 + c^2v^2 - c^2u^2}{\rho v(u^2 - v^2)}\,\frac{1}{ik}\,\frac{\partial\rho'}{\partial x}; \\
u'_x &= u\,\frac{u^2v^2 + c^2v^2 - c^2u^2}{\rho v^2(u^2 - v^2)}\,\frac{1}{ik}\,\frac{\partial\rho'}{\partial x}.
\end{aligned}\right\} \tag{187}$$

The condition of absence of normal component of velocity, in accordance with equation (73), should be fulfilled on the stationary boundary surface:

$$v'_x - ikv\xi = 0 \quad \text{when} \quad x = 0. \tag{188}$$

The above condition gives one relationship between the two unknown coefficients C_1 and C_2 which appear in the solution of the problem. From expressions (186) to (188), we find:

$$C_1 - C_2 = \frac{k\varkappa M^2}{1 - M^2}\rho\xi_0. \tag{189}$$

The second condition for the above coefficients is easily found, when the flow is of the elliptical type. In this case, $\varkappa > 0$ and for a limiting perturbation with $x = \infty$ it should be assumed that $C_1 = 0$. Thus,

$$C_2 = -\frac{k\varkappa M^2}{1 - M^2}\rho\xi_0. \tag{190}$$

For this case the perturbation of full pressure will be determined by:

$$P' \equiv c^2\rho^1 + \rho u u'_y = \frac{u^2 - v^2}{\varkappa} k\rho\xi_0 e^{-k\varkappa x + iky}. \tag{191}$$

The above expressions, analogous with ordinary hydrodynamics, permit the explanation of the nature of instability of tangential disturbances. Indeed, assuming $x = 0$, a full pressure perturbation on the boundary surface may be found:

$$P' = \frac{u^2 - v^2}{\varkappa}\rho k\xi(y). \tag{192}$$

It follows that with $u^2 < v^2$, change of full pressure at the boundary takes place in the opposite direction or out of phase in relationship to the disturbance of the surface. For example, where the disturbance is positive the pressure will be lowered as shown in Figure 10. The above produces a force which tends to increase the disturbance of the surface. If the surface is stationary, as for example in the case of a tangential disturbance, then a small perturbation of a flat surface will be increased. Such an effect has a purely hydrodynamic origin and derives from the fact that in places of narrowing of the stream, the velocity increases and the pressure falls correspondingly. Conversely, with $u^2 > v^2$, as shown in equation (192), the change of full pressure on the boundary surface is such that a force is produced which tends to straighten out the surface. This follows from the fact that with a sufficiently strong field $(H/\sqrt{4\pi\rho} > v)$, a hydrodynamic lessening of pressure in places of narrowing of the stream is overshadowed by the rise of the magnetic pressure. The above is a description of elliptical flows where $(\varkappa^2 > 0)$.

For hyperbolic flows, $\varkappa = -i\lambda$, where λ is a real number, and both terms of equation (186) are limited by $x = \infty$. With conditions of perturbations where $(\lambda > 0)$, i.e., for perturbations that occur in the same direction of flow as that of the liquid, the full pressure perturbation is determined by the following expression:

$$P' = -i\frac{u^2 - v^2}{\lambda}\rho k\xi_0 e^{ik(y - \lambda x)}. \tag{193}$$

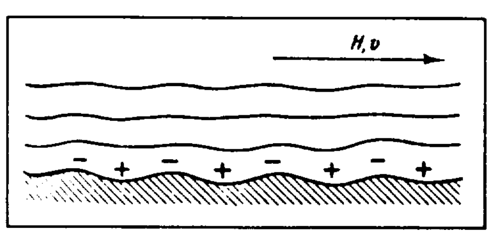

Fig. 10. Elliptical flow.

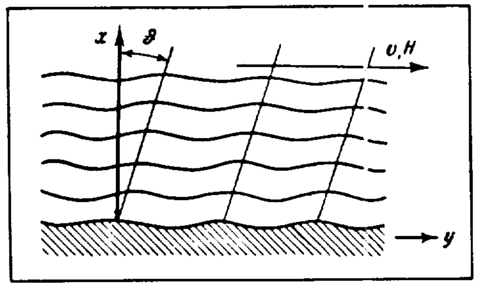

Fig. 11. Hyperbolic flow.

In this case, the change of full pressure on the boundary is out of phase by $\pi/2$ with the displacement of the surface. The amplitude of the perturbation remains constant in space. Equal phase lines are inclined to the normal at an angle ϑ, where $\operatorname{tg}\vartheta = dy/dx = \lambda$, as shown in Figure 11. This means that with hyperbolic flow of a compressible conductive medium in a magnetic field, a perturbation originating from some point is propagated only along the direction of the flow inside a cone with $\alpha = 90° - \vartheta$, and where $\operatorname{tg}\alpha = 1/\lambda$.

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SECTION 9. STEADY STATE MOTION OF AN IDEAL MEDIUM

With steady state motion $(\partial / \partial t = 0)$, the system of magnetohydrodynamic equations (24) for a medium whose conductivity may be considered infinite and whose viscosity and thermal conductivity may be neglected reduces to:

$$\operatorname{rot}[\boldsymbol{vH}] = 0; \tag{194}$$

$$\operatorname{div}\boldsymbol{H} = 0; \tag{195}$$

$$\frac{1}{\rho}\nabla p + \frac{1}{4\pi\rho}[\boldsymbol{H}\operatorname{rot}\boldsymbol{H}] - [\boldsymbol{v}\operatorname{rot}\boldsymbol{v}] - \frac{1}{2}\nabla \boldsymbol{v}^2 = 0, \tag{196}$$

$$\operatorname{div}\rho\boldsymbol{v} = 0; \tag{197}$$

$$\boldsymbol{v}\nabla s = 0. \tag{198}$$

Equation (194) is equivalent to the condition:

$$[\boldsymbol{vH}] = \operatorname{grad}\varphi, \tag{199}$$

where φ is an arbitrary function of the coordinate system. Since with an infinite conductivity $\boldsymbol{E} = -\frac{1}{c'}[\boldsymbol{vH}]$ in according with equation (30), then the electric field should be devoid of eddies: $\boldsymbol{E} = -\operatorname{grad} c'\varphi$. Obviously, $c'\varphi$ is simply the potential of the electric field. From condition (199), vectors $\boldsymbol{v}$ and $\boldsymbol{H}$ in steady state motion should be perpendicular to the gradient of the potential, i.e., vector lines $\boldsymbol{v}$ and $\boldsymbol{H}$ should lie on equipotential surfaces of the electric field.

First the flow of an incompressible liquid will be investigated. From equations (194) to (198) it follows that:

$$\operatorname{rot}[\boldsymbol{vH}] = 0; \tag{200}$$

$$\operatorname{div}\boldsymbol{H} = 0; \tag{201}$$

$$\operatorname{div}\boldsymbol{v} = 0; \tag{202}$$

$$\frac{1}{\rho}\nabla p + \frac{1}{4\pi\rho}[\boldsymbol{H}\operatorname{rot}\boldsymbol{H}] - [\boldsymbol{v}\operatorname{rot}\boldsymbol{v}] + \nabla\frac{\boldsymbol{v}^2}{2} = 0. \tag{203}$$

Exclusion of pressure from the above system of equations will yield:

$$\mathrm{rot}\left\{[\boldsymbol{v}\,\mathrm{rot}\,\boldsymbol{v}] - \frac{1}{4\pi\rho}[\boldsymbol{H}\,\mathrm{rot}\,\boldsymbol{H}]\right\} = 0. \tag{204}$$

Equations (200) to (202) and (204) determines $\boldsymbol{v}$ and $\boldsymbol{H}$ as a function of coordinates. The solution for the above is:

$$\boldsymbol{v} = \pm\frac{\boldsymbol{H}}{\sqrt{4\pi\rho}}, \tag{205}$$

where one of the quantities may be an arbitrary function of the coordinates and satisfies condition (201) or (202). The pressure is determined from equation (203), or by the use of equation (205), from the following:

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$$\nabla\left(p + \frac{H^2}{8\pi}\right) = 0, \tag{206}$$

which shows that along the entire space,

$$p + \frac{H^2}{8\pi} = \mathrm{const.} \tag{207}$$

Equation (205) is an exact solution of magnetohydrodynamic equations which correspond to steady state motion of a medium along an arbitrary magnetic field with a velocity that depends on the intensity of the field. It is convenient to represent such motion by "tubes" of current which correspond to the force "tubes" of the magnetic field. The interaction of neighboring tubes takes place through transverse pressure $P = p + H^2/8\pi$. There are possible surfaces of tangential disturbances upon which all quantities may undergo arbitrary step-changes as long as condition (207) is fulfilled. In particular, flow of any form may be realized as long as it is limited by the surface of the tangential disturbance of the field, velocity, and, in a general case, density. Applying the criterion of stability (107) to such a discontinuous solution, the condition of stability according to equation (205) will assume the following form:

$$[\sqrt{\rho_1}(\boldsymbol{k}_0\boldsymbol{H}_1) + \sqrt{\rho_2}\,(\boldsymbol{k}_0\boldsymbol{H}_2)]^2 \geqslant 0\,. \tag{208}$$

Thus, dynamically stable steady state flows of a conductive medium along the force lines of an arbitrary magnetic field are possible in the form of separate jets or streams. From the point of view of ordinary hydrodynamics, such motion would be impossible for two reasons. First, the motion of a stream in ordinary hydrodynamics, with the absence of any external forces, is along a straight line. Secondly, due to the absolute instability of tangential disturbances, the stream will rapidly become turbulent and will mix in the ambient fluid.

In the solution of equation (205), the magnetic field and the velocity of the medium are related by the same correspondence which is found in the perturbation quantities of the wave of Ref. [7] which is propagated along an external magnetic field. In cases where the density both inside and outside of the stream is equal, then the solution to equation (205) may be considered as a limiting case in a magnetohydrodynamic wave with the absence of an external magnetic field. The same conclusions follow from the results of Section 3, above. Namely, with a continuous density, a disturbance on both sides of which the condition (205) is fulfilled is a transition between a tangential disturbance and a magnetohydrodynamic wave. Also, by difference with the magnetohydrodynamic wave, the solution of equation (205) may describe a motion where the density goes through a discontinuity: for example, motion of a stream in a medium of a different density. In addition, the solution to equation (205) can be applied, under some other conditions, to a compressible medium.

The steady state motion of a compressible medium takes place according to equations (194) to (198). The conditions under which these equations permit a solution in the form of equation (205) are here established. Such a solution always satisfies equation (194). Substituting equation (205) in equation (197) yields:

$$\operatorname{div}(\boldsymbol{H}\sqrt{\rho}) = \sqrt{\rho}\operatorname{div}\boldsymbol{H} + \frac{\boldsymbol{H}}{2\sqrt{\rho}}\nabla\rho = 0.$$

Since $\operatorname{div}\boldsymbol{H} = 0$, then

$$\boldsymbol{H}\nabla\rho = 0, \tag{209}$$

i.e., the density of the medium should remain constant along force lines of the magnetic field. Taken with equation (198) this denotes that along the force lines all remaining thermodynamic functions should be constant:

$$\boldsymbol{H}\nabla p = 0. \tag{210}$$

Also, equation (196) will be reduced to the condition:

$$\nabla\left(p + \frac{H^2}{8\pi}\right) = 0 \tag{211}$$

or, by considering equation (210):

$$\boldsymbol{H}\nabla H^2 = 0. \tag{212}$$

It may be deducted from the above that the intensity of the magnetic field does not change along the force tubes, that the cross section of force tubes is constant, and that the motion along them takes place with a constant velocity. Thus, in a compressible medium, the solution to equation (205) may be realized in the form of uniform motion of the medium along force tubes of the magnetic field, the cross section of which does not change. All remaining conclusions, including the possibility of occurrence of tangential disturbances, remain the same as for an incompressible medium.

SECTION 10. POSSIBLE ASTROPHYSICAL APPLICATIONS

At the present time, the information available on the dynamics of an ionized gas in a magnetic field is quite limited. According to the evaluations of Ref. [9], magnetohydrodynamic effects should play an important role in the dynamics of interstellar gas, stellar atmospheres, and in particular the atmosphere of the sun. The investigation of motion of interstellar gas masses is made difficult by the fact that notable changes in these objects take place very slowly, which does not permit the gathering of a sufficient quantity of data. On the other hand, the atmosphere of the sun presents a different picture. Here, the motions develop relatively fast, giving rise to the possibility of gathering sufficient data during relatively short periods of time and of observing the character of these motions. In this respect, most promising are investigations of solar protuberances, which are characterized by gaseous formations on the boundary of the chromosphere and the corona, and which in turn result from the activity of the sun.

The phenomenon of protuberances is very complicated since it develops in a gravitational field in the region of large non-uniformities of temperature and density. It is enough to say that the adopted classification of protuberances on the basis of their external signs and solar spots (Ref. [22]) includes six classes and seventeen sub-classes. However, as shown by Refs. [23] and [24], three basic classes may be distinguished in the protuberances: 1) eruptive; 2) orderly or electromagnetic; and 3) chaotic or turbulent. Detailed description of these classes shows that the dynamics of protuberances cannot be explained from the point of view of ordinary hydrodynamic concepts. Therefore, a hypothesis was proposed to determine the function of the electromagnetic field. This was done in Refs. [23] and [24]. The arguments in favor of this hypothesis are as follows:

1) trajectories of the protuberances quite often have a regular form which resembles the picture of force lines of a magnetic pole;

2) protuberances are closely related to sunspots, which possess strong magnetic fields;

3) complex forms of protuberances remain for extended periods of time in an equilibrium which does not appear to be hydrostatic.

The electromagnetic hypothesis has been made in a quite general form and there are no concrete results. It is shown below that the use of this hypothesis in the framework of magnetohydrodynamics permits the explanation of some characteristic motions of protuberances and the evaluation of related magnetic fields.

At the present time, there is an absence of data permitting any evaluation of the sources which cause the appearance of protuberances. The formation of protuberances is closely related to solar activity and apparently may be explained by processes taking place in the inner regions of the sun. Therefore, the discussion will be limited to the evaluation of some characteristics of motion of the protuberances without the investigation of their sources.

Since protuberances develop in those layers of the sun's atmosphere possessing high conductivity ($\delta \approx 10^{13}$) and, in addition, are related to solar spots, which in turn possess strong magnetic fields (up to 3500 oersteds), it may be expected that their motion occurs in accordance with the laws of magnetohydrodynamics.

The protuberances of the second class are easiest to interpret in terms of the "magnetic hypothesis". Such protuberances possess the following properties as listed by Ref. [24]:

1) the motion of matter takes place longitudinally along curved, discreet trajectories that are fixed in space;

2) the direction of the motion depends only on the distribution of such trajectories and does not have a direct relationship to the upward or downward direction relative to the surface of the sun;

3) such trajectories exist for prolonged periods of time without noticeable changes, and the motion along them is repeated;

4) in most cases the motion along such trajectories is uniform;

5) the cross section of streams and jets is approximately uniform along their length.

These properties correspond to the properties found in the last section for a steady state solution of magnetohydrodynamic equations. Actually, in compressible medium, such as the solar atmosphere, the solution

$$\boldsymbol{v} = \pm \frac{\boldsymbol{H}}{\sqrt{4\pi\rho}} \tag{213}$$

exists, if the density and the absolute values of the magnetic field intensity and of the velocity do not change along the force lines. This means that the cross section of the stream, corresponding in its form to equation (213) and corresponding to the force tube of the magnetic field, remains unchanged and the motion is uniform. This, in turn, corresponds to the properties of the investigated protuberances. The possibility of analyzing this as a steady state phenomenon follows from the above-mentioned property 3) of repeatability of motion along the same trajectories.

It will be noted that equation (213) shows only general properties of the dynamics of protuberances. Such phenomena as burnout of protuberances by radiation of the corona, their luminescence, etc. are not considered here. The latter are secondary effects, related to the displacement of matter from one layer of the solar atmosphere to another. It is important to note that calculations of the velocity of protuberances that are based upon measurement of velocity of nodes and other luminescent details may not correspond to the actual velocities of the matter, in cases where the process of illumination is propagated along the trajectory with a velocity different from the velocity of the matter.

In real conditions, the requirement that the density and intensity of the magnetic field should be constant along the stream is too severe. However, it may be expected that small changes in this condition will not radically change the character of the motion.

Expression (213) permits the evaluation of the magnetic field intensity related to protuberances of the investigated class. Assuming the average density of matter in protuberances to be 10^{-14} gm/cm^3, it will be found that, for a steady state motion of matter along force lines of magnetic field with observed velocity of 30 to 300 km/sec, the intensity should be equal to 1 to 10 oersteds. It follows from the observations of Ref. [25], that the surface of the sun contains local fluctuations of the magnetic field that reach 30 oersteds. It is possible that these chaotic magnetic fields are related to the motion of the protuberances.

An important property of solutions (213) for streams or jets is the stability of the surface separating the stream from the remainder of the gaseous masses during abrupt changes of velocity. Since such a surface may be considered a surface of a tangential disturbance, its stability follows from the results of Sections 5 and 6. The stabilizing action of the magnetic field, explains the fact that jets and streams observed in the protuberances conserve their form for a long period of time. From the point of view of ordinary hydrodynamics, however, they should be transformed into turbulent streams in a matter of a short time and should mix with the ambient medium.

Thus the steady state solution of magnetohydrodynamic equation (213), taken with the results of investigations of the stability of tangential disturbances permits the explanation of a number of characteristic features of the dynamics of protuberances. Until recently, only protuberances of the second class were investigated, due to the regular, flow-like motion of matter. In correspondence with the above, it may be assumed that protuberances of the third class are chaotic or turbulent and develop in a region of weak magnetic field. Their motion, in accordance with ordinary hydrodynamics, should have turbulent character. The properties of eruptive protuberances are probably determined by the conditions of their appearance, which are not investigated here.

CONCLUSIONS

The investigation of magnetohydrodynamic disturbances presented in this work leads to the following basic results:

1. There are four types of disturbances in magnetohydrodynamics. These depend on the character of step-changes of velocity and intensity of the field on the surface of the disturbance: tangential, perpendicular shock wave, inclined shock wave and magnetohydrodynamic wave.

2. All types of magnetohydrodynamic disturbances are related by mutual transitions, so that with a continuous change of conditions of motion, a disturbance of one type may be transformed into a disturbance of another type. Such transitions may take place as a result of small perturbations of the surface.

3. A criterion has been obtained which determines the stability of tangential disturbances in an incompressible medium. This criterion shows that a sufficiently strong magnetic field, parallel to the direction of the motion of the medium, stabilizes the tangential disturbance.

4. Regions of parameters have been found that characterize tangential velocity disturbances in a compressible medium with the presence of magnetic field. In such cases the surface disturbance is stable.

5. In both compressible and incompressible media, the stability of motion takes place when the density of the magnetic energy reaches a value equal to the density of the kinetic energy of the relative motion of the medium.

6. An expression for the width of a perpendicular shock wave of low intensity has been obtained. This expression shows that low conductivity of the medium may lead to a significant widening of the shock wave in a strong magnetic field.

7. A problem on the flow around small obstacles in magnetohydrodynamics was investigated. It was shown that the character of propagation of perturbations in steady state flow in a compressible medium is determined by two dimensionless parameters: v/c and $H/c\sqrt{4\pi\rho}$. Depending upon the values of these dimensionless parameters, the flow may be classified as either the elliptical or the hyperbolic type.

8. It was shown that magnetohydrodynamic equations for incompressible media permit an exact solution of steady state equations of the form: $\boldsymbol{v} = \pm \boldsymbol{H}/\sqrt{4\pi\rho}$, where $\rho = \text{const}$ and $\boldsymbol{H}$ is an arbitrary magnetic field. Such solutions exist for a compressible medium with the conditions $\boldsymbol{H}\nabla\rho = 0$ and $\boldsymbol{H}\nabla H^2 = 0$.

9. The steady state solutions to magnetohydrodynamic equations taken with the results of investigation of stability of tangential disturbances permit the explanation of some characteristics of motion in the solar atmosphere, i.e., regular motion in solar protuberances.

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